

SOME MATHEMATICAL PRINCIPLES OF MILITARY COMBAT AND COUNTER - INSURGENCY

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1.0 INTRODUCTION

The art of war has evolved over the years into strategic engagements where quantitative use of calculations and mathematical models will be an advantageous asset. Global conflicts in recent years involve clashes between guerrilla armies, counter-insurgences and conventional forces. It is possible to use mathematical models to plausibly depict combat scenarios to a greater extent. The role of the mathematical model is to help military battle field commanders to perform the following tasks:

- (I) Use mathematics to advantageously relate combat variables quantitatively.
- (II) Enables the derivation of robust theoretical criteria for therapeutic outcome of victory, stalemate and defeat.
- (III) Facilitate the use of computers to give detailed predictions and dynamic view of battle.
- (IV) Provides alternative templates to counteract the moves of the opponents.
- (V) Enables the battlefield commander to avoid disastrous gambles and catastrophic mistakes.

There have been many attempts in combat modeling in the literature. [Hofbauer and Sigmund 1991; Przemieniecki 2000; Narayan and Miller 2002]. Some of such models involve the use of ODE, PDE, Stochastic Differential Equations, Discrete Differential Equations and Markovian Processes. The most common models involve descriptive, deterministic, time continuous and state continuous types of models involving ODE. In the literature the most popular model used are the Lanchester models.

1.1 Models of Warfare

In this section, previous work on models depicting military combat will be issued. In particular, their advantage and shortcoming will be explained elaborately.

1.1.1 The Lanchester Guerrilla Warfare Model

This model is also called the Lanchester's linear law (Un-directed Fired Model).

The system of ODE describing this model is:

$$\begin{cases} \frac{dx_1}{dt} = -a_{12}x_1x_2 \\ \frac{dx_2}{dt} = -a_{21}x_1x_2 \end{cases}$$

$$x_1(t=0) = x_{10}$$

$$x_2(t=0) = x_{20}$$

where x_1 = guerrilla army #1

x_2 = guerrilla army #2

a_{12} = The specific inter-army killing rate, constant. The number of soldiers of army # 1 killed by a single soldier of army # 2 per unit time during combat engagement with army # 2

a_{21} = The specific inter-army killing rate, constant. The number of soldiers of army # 2 killed by a single soldier of army # 1 per unit time during combat engagement with army # 1

This model is excellent, but it has some shortcomings

- (i) There are no terms for reinforcement of the armies. Thus the model depicts what can be characterized as “fight to the death”.
- (ii) There are no terms representing troop loss or death due to non-combat cases such as desertion, suicide or death to natural causes.
- (iii) This model applies to battle field conditions such as two armies fight in a closed battle field with no re-enforcements coming to either side. The battle will then proceed until one army is annihilated or surrenders. It is possible for both armies to fight to the death.

Other examples of the use of mathematical models in military combat found in the literature include Mathematical modeling of the Battle of Agincourt, 1415 [Eggenberger 2007], Kolmogorov’s equations of combat. In this example the mathematical model consist of a system of partial differential equations.

In this thesis mathematical models of military combat will be constructed which are modifications and improvements on the Lanchester model. The improved models will be in general applied to counter-insurgences as well.

1.2 Mathematical Interpretation of the Lanchester Guerrilla Model (LGM)

In this section, some solutions of the LGM model will be presented.

1.2.1 The Linear Law of Undirected Fire

The Linear law of Undirected Fire describes firing that is simply directed into the general area where the enemy's units of combat are located under the next characteristics:

- Both forces are homogeneous and are continually engaged in combat
- Each force is within the maximum range of all the opposing forces.
- Each force is aware only of the general area of location of the enemy forces

The next system of differential equations represents this model:

$$\dot{x}_1 = -a_{12}x_1x_2$$

$$\dot{x}_2 = -a_{21}x_1x_2$$

The battle outcomes for the model are:

$$O_1[0,0] \quad (\text{Both armies fight to mutual annihilation})$$

$$O_2[0, n_2] \quad (\text{Army \# 1 is exterminated by army \# 2})$$

$$O_1[n_1, 0] \quad (\text{Army \# 2 is exterminated by army \# 1})$$

1.2.2 Interpretations of the Law of Undirected Fire

Now consider the LGM using a revised different approach:

$$\begin{cases} \dot{x}_1 = -a_{12}x_1x_2 \\ \dot{x}_2 = -a_{21}x_1x_2 \end{cases}$$

$$\frac{\dot{x}_1}{\dot{x}_2} = \frac{-a_{12}x_1x_2}{-a_{21}x_1x_2} = +\frac{a_{12}}{a_{21}}$$

$$\Rightarrow \frac{\dot{x}_1}{\dot{x}_2} = \frac{a_{12}}{a_{21}}$$

$$a_{21}\dot{x}_1 = a_{12}\dot{x}_2$$

$$a_{21}x_1 + x_{10} = a_{12}x_2 + x_{20}$$

$$a_{21}x_1 - a_{12}x_2 = x_{20} - x_{10} = k$$

Consider the following scenarios.

Scenario # 1

Suppose $x_{20} = x_{10}$ and $a_{21} = a_{12}$

In this case the battle may end in a stalemate as both armies fight to a standstill with both sides losing troops at the same rate. Figure 1.1 shows the dependence between x_1 and x_2 for this scenario.

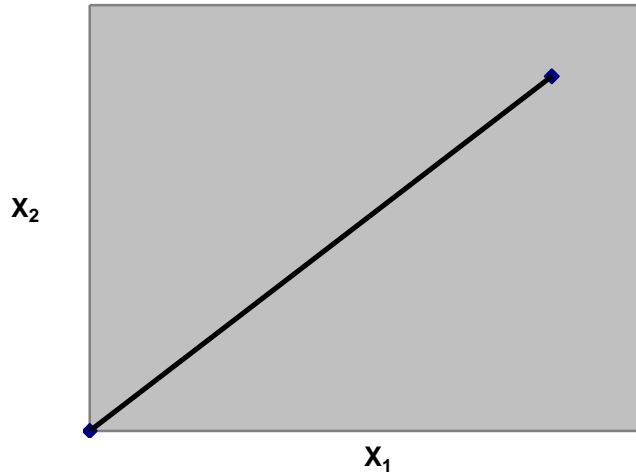


FIGURE 1.1 Graph of Scenario #1

Scenario # 2

Suppose $x_{20} = x_{10}$ and $a_{12} > a_{21}$. Then $x_2 = \frac{a_{21}}{a_{12}} x_1$,

But x_1 and x_2 are inversely related according to the model equations and definitions.

Thus if $a_{12} > a_{21}$, then x_1 decreases as x_2 increases.

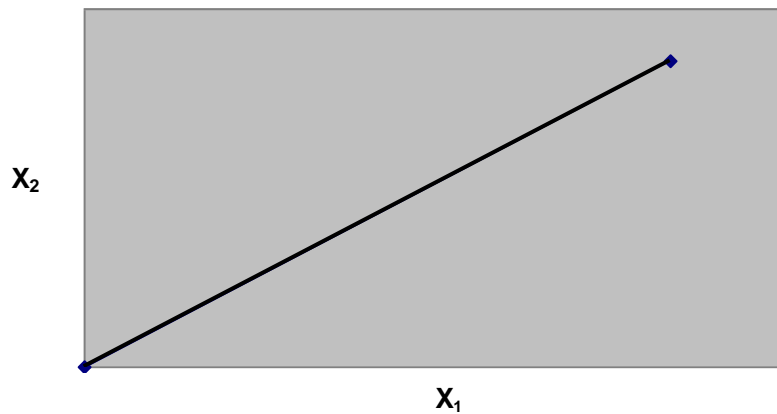


FIGURE 1.2 Graph of Scenario #2

Scenario # 3

Suppose $x_{20} = x_{10}$ and $a_{12} < a_{21}$

Then from $x_2 = \frac{a_{21}}{a_{12}} x_1$, it can be concluded that x_2 will decrease due to a similar argument

involving an increase in x_1 , provided by a decrease in a_{12} .

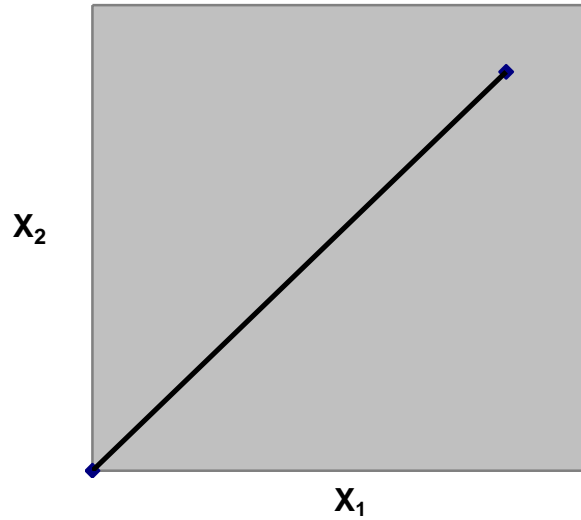


FIGURE 1.3 Graph of Scenario #3

Scenario # 4

Suppose $k = x_{20} - x_{10} \neq 0$

Then x_1 and x_2 are linked by the equation:

$$x_2 = \frac{a_{21}}{a_{12}} x_1 - \frac{k}{a_{12}}$$

So,

$$x_2 = \frac{a_{21}}{a_{12}} x_1 - \frac{x_{20} - x_{10}}{a_{12}}$$

2.0 Simple Generalizations of the Lanchester Warfare

In this chapter simple generalizations of Lanchester's warfare models will be done. In addition, mathematical models depicting warfare between more than two armies will be constructed. These generalized models will be analyzed using dynamical systems theory and the principals of linearized stability, Hartman- Grobman theorem and Jacobian matrix analysis [Nani 2002].

2.0.1 M1 Model

Definition of Parameters and Constants

x_1 : The number (cardinality) at time t of soldiers of army #1

x_2 : The number (cardinality) at time t of soldiers of army #2

a_{12} = The specific inter-army killing rate constant. The number of soldiers of army # 1 killed per unit time by a single soldier of army # 2 during combat engagement with army # 2.

a_{21} = The specific inter-army killing rate constant. The number of soldiers of army # 2 killed per unit time by a single soldier of army # 1 during combat engagement with army # 1.

This model represents a closed battle field, i.e. a battlefield where no troop supplies are available for the fighting armies. The model is given by the next system of equations:

$$\begin{cases} \dot{x}_1 = -a_{12}x_1x_2 \\ \dot{x}_2 = -a_{21}x_1x_2 \\ x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \end{cases}$$

After finding the Equilibrium points of the systems, it can be conclude that the equilibrium points of the system are always located in the axes and are of the type:

$E_1 = [0,0]$ (Both armies fight to mutual annihilation)

$E_2 = [k_1,0]$ (Army # 2 is annihilated by army # 1)

$E_3 = [0,k_2]$ (Army # 1 is annihilated by army # 2)

2.0.2 M2 Model

Model two represents an open battlefield. In an open battlefield the fighting armies receive reinforcements during the conflict. Thus considering the supply of troops, the Lanchester model becomes:

Definition of Parameters and Constants

x_1 : The number (cardinality) at time t of soldiers of army # 1

x_2 : The number (cardinality) at time t of soldiers of army # 2

a_{12} : The specific inter-army killing rate constant. The number of soldiers of army # 1

killed per unit time by a single soldier of army # 2 during combat engagement with army # 2.

a_{21} = The specific inter-army killing rate constant. The number of soldiers of army # 2 killed per unit time by a single soldier of army # 1 during combat engagement with army # 1.

S_1 : Rate of reinforcement of army # 1 at a time t . The rate of change of the number of Soldiers of army # 1 sent per unit of time to the battle field as reinforcement.

S_2 : Rate of reinforcement of army # 2 at a time t . The rate of change of the number soldiers of army # 2 sent per unit of time to the battle field as reinforcement.

$$\begin{cases} \dot{x}_1 = S_1 - a_{12}x_1x_2 \\ \dot{x}_2 = S_2 - a_{21}x_1x_2 \\ x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \end{cases}$$

where the next supplies conditions are not feasible:

$$S_1 = 0 \text{ and } S_2 \neq 0$$

$$S_1 \neq 0 \text{ and } S_2 = 0$$

The only feasible solution for the rest point can be found when,

$S_1 \neq 0$ and $S_2 \neq 0$. In this case the rest point will be an interior point $E = [\hat{x}_1, \hat{x}_2]$.

2.0.3 M3 Model

Model three is the more realistic representation of a battlefield. It takes into consideration key factors as desertion, death by epidemics and non-combat issues such as accidents and friendly fire. These factors play a decisive role in the outcome of combat and make the model more accurate and real. Thus, by considering the factors mentioned previously the Lanchester model now becomes:

$$\begin{cases} \dot{x}_1 = S_1 - a_{12}x_1x_2 - a_1x_1 - k_{12}x_2 \\ \dot{x}_2 = S_2 - a_{21}x_1x_2 - a_2x_2 - k_{21}x_1 \\ x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \end{cases}$$

where a_i : defection death rate constant by epidemic, friendly fire and by noncombat issues of army # i , $i = \{1,2,3\}$.

The constant k_{12} denotes the rate of decrease of army #1 due to events such as suicide bombing or land mines created by army # 2. Similarly the constant k_{21} denotes the rate of decrease of army # 2 due to suicide bombings or land mines created by army # 1.

To find the equilibrium point or rest points of the system we solve the system for $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

The system has at most three rest long term battle outcomes. To find them we proceed as follows:

Let $x_1 = 0$ in the first equation and second equations of the system. Then,

$$S_1 - k_{12}x_2 = 0 \quad \text{and} \quad S_2 - a_2x_2 = 0$$

Thus,

$$\frac{S_1}{S_2} = \frac{k_{12}}{a_2}$$

Let $x_2 = 0$ in the first equation and second equations of the system. Then,

$$S_1 - a_1 x_1 = 0 \text{ and } S_2 - k_{21} x_1 = 0$$

$$\text{Hence } x_1 = \frac{S_1}{a_1} = \frac{S_2}{k_{21}}$$

Thus,

$$\frac{S_1}{S_2} = \frac{a_1}{k_{21}}$$

Therefore,

$$\frac{S_1}{S_2} = \frac{k_{12}}{a_2} = \frac{a_1}{k_{21}}$$

But,

$$S_1 - S_2 - k_{12} x_2 + a_2 x_2 = 0$$

$$(a_2 - k_{12}) x_2 = S_2 - S_1$$

$$x_2 = \frac{S_2 - S_1}{a_2 - k_{12}}$$

$$x_2 = \frac{S_1 - S_2}{k_{12} - a_2}$$

$$\text{Hence } E_1 = \left[0, \frac{S_1 - S_2}{k_{12} - a_2} \right]$$

where $k_{12} > a_2$ and $S_1 > S_2$ or $k_{12} < a_2$ and $S_1 < S_2$

Due to The reversing order of equations x_1 can be obtained directly.

Thus,

$$x_1 = \frac{S_2 - S_1}{k_{21} - a_1}$$

Hence $E_2 = \left[\frac{S_2 - S_1}{k_{21} - a_1}, 0 \right]$

Where $k_{21} > a_1$ and $S_2 > S_1$ or $k_{21} < a_1$ and $S_2 < S_1$

The third equilibrium point $E_3 = [X_1, X_2]$ exists if the following two surfaces:

$$\Gamma_1 : S_1 = a_{12}x_1x_2 + a_1x_1 + k_{12}x_2$$

$$\Gamma_2 : S_2 = a_{21}x_1x_2 + a_2x_2 + k_{21}x_1$$

intersect at the point $[X_1, X_2]$

Hence $E_3 = [X_1, X_2]$ depicts the interior equilibrium, with the following specifications

showed in the table below,

X_1	X_2
0	$\frac{S_1 - S_2}{k_{12} - a_2}$
$\frac{S_2 - S_1}{k_{21} - a_1}$	0
\hat{x}_1	\hat{x}_2

TABLE 1.1 Long term battle outcomes

2.1 Generalized Combat Mathematical Models

In this section the Lanchester model will be generalized to include more than two combating armies. The model also will apply to guerilla armies or insurgencies who are fighting an occupation by an elite army. In this case the battlefield is asymmetrical due to the non-conventional fighting techniques by one or more of the combating armies.

Mathematical models of combat can be used to understand what factors can influence the outcome of the battle: some questions which might be asked include which side is the victor, how many survivors remain, how long does the battle take?

The two mathematical models used in the work make use of Lanchester [] but substantially improve the model by far when taking into consideration key factors that determine the outcome of combat. The models are the surge supply model and the logistic supply model. Both models take into consideration reinforcement of the army and gives a very representative picture of real combat situations. The models are governed by three simultaneous ordinary differential equations (ODE.) The equations include terms that represent reinforcement, troop loss due to inter army combat, troop loss due to non-combat cases, and troop loss due to recruitment from one army to another. By considering all those terms, the obtained model will become more realistic and precise.

Definition of Parameters

The following is the notation that will be used to define the parameters of the ODE,

S_1 : Rate of reinforcement of army # 1 at a time t . The rate of change of the number of soldiers of army # 1 sent per unit of time to the battle field as reinforcement.

S_2 : Rate of reinforcement of army # 2 at a time t . The rate of change of the number soldiers of army # 2 sent per unit of time to the battle field as reinforcement.

x_1 : Number (cardinality) of soldiers of invading army # 1 at time t .

x_2 : Number of resistance (insurgent) indigenous fighters at time t .

x_3 : Number of fighters of the invading guerilla army opposed to both invading army one and indigenous army two at time t .

Due to tribal affiliation some recruitment from army two to army three occurs.

Specifications of Constants

a_1, b_1 : Surge term constants which determine the peak and spread values of surge respectively.

$K_{i,j}$: Specific casualty death rate due to inter army combat, where $i, j \in \{1,2,3\}$

k_i : Specific death rate due to non-combat actions, where $i \in \{1,2,3\}$

$r_{i,j}$: Recruitment rate of fighters from army i to army j

The next diagram gives an illustration of the battle field scenario and the interaction of the fighting groups.

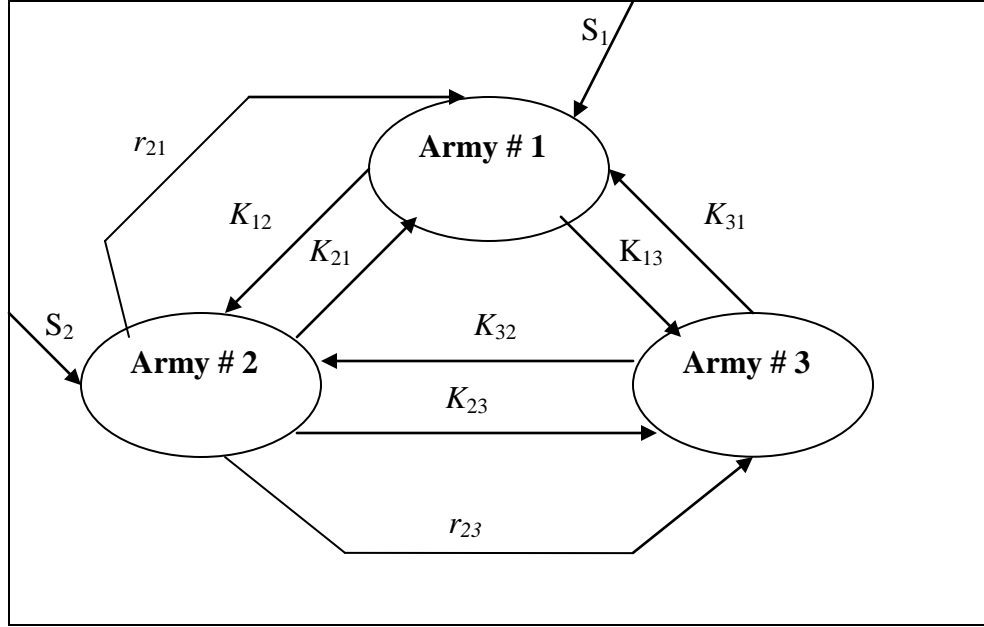


FIGURE 2.1 Schemata of the Conflict

The general form of the mathematical model describing the conflict is given by the next system of ODE:

$$\dot{x}_1 = S_1 - k_{12}x_1x_2 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2$$

$$\dot{x}_2 = S_2 - k_{21}x_2x_1 - r_{23}x_2 - k_{23}x_2x_3 - k_2x_2 - k_{22}x_2^2 - r_{21}x_2$$

$$\dot{x}_3 = r_{23}x_2 - k_{31}x_3x_1 - k_{32}x_3x_2 - k_3x_3 - k_{33}x_3^2$$

2.2 Generalized Combat Model with Surge Source

In this section a generalized combat model with logistic source is discussed. In particular, the army #1 has a logistic reinforcement. The model equations are displayed as follows:

Here, the surge function is given by:

$$f(x) = S_1 = a_1 x_1 e^{-b_1 x}$$

S_1 : Rate of reinforcement of army # 1 at a time t. The rate of change of the number of soldiers of army # 1 sent per unit of time to the battle field as reinforcement.

S_2 : Rate of reinforcement of army # 2 at a time t. The rate of change of the number of soldiers of army # 2 sent per unit of time to the battle field as reinforcement.

x_1 : Number (cardinality) of soldiers of invading army # 1 at time t.

x_2 : Number of resistance (insurgent) indigenous fighters at time t.

x_3 : Number of fighters of the invading guerilla army opposed to both invading army one and indigenous army two at time t.

Due to tribal affiliation some recruitment from army two to army three occurs.

Specifications of Constants

a_1, b_1 : Surge term constants which determine the peak and spread values of surge respectively.

$K_{i,j}$: Specific casualty death rate due to inter army combat, where $i, j \in \{1,2,3\}$

k_i : Specific death rate due to non-combat actions, where $i \in \{1,2,3\}$

$r_{i,j}$: Recruitment rate of fighters from army i to army j

$$\begin{cases} \dot{x}_1 = a_1 x_1 e^{-b_1 x_1} - k_{12} x_1 x_2 - k_{13} x_1 x_3 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2 \\ \dot{x}_2 = S_2 - k_{21} x_2 x_1 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 \\ \dot{x}_3 = r_{23} x_2 - k_{31} x_3 x_1 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2 \end{cases}$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases}$$

where a_1, b_1 are the surge term constants which determine the peak and spread values of the surge respectively.

2.3 Generalized Combat Model with Logistic Source

In this section a generalized combat model with logistic source will be discussed. In this case the logistic function is represented by the expression:

$$f(x) = S_1 = (a_1 - b_1 x_1) x_1$$

S_1 : Rate of reinforcement of army # 1 at a time t . The rate of change of the number of soldiers of army # 1 sent per unit of time to the battle field as reinforcement.

S_2 : Rate of reinforcement of army # 2 at a time t . The rate of change of the number soldiers of army # 2 sent per unit of time to the battle field as reinforcement.

x_1 : Number (cardinality) of soldiers of invading army # 1 at time t .

x_2 : Number of resistance (insurgent) indigenous fighters at time t .

x_3 : Number of fighters of the invading guerilla army opposed to both invading army one and indigenous army two at time t .

Due to tribal affiliation some recruitment from army two to army three occurs.

Specifications of Constants

a_1, b_1 : Surge term constants which determine the peak and spread values of surge respectively.

$K_{i,j}$: Specific casualty death rate due to inter army combat, where $i, j \in \{1,2,3\}$

k_i : Specific death rate due to non-combat actions, where $i \in \{1, 2, 3\}$

$r_{i,j}$: Recruitment rate of fighters from army i to army j

Model Equations

The model equations are presented as follows. In particular, these equations have been constructed and analyzed under slightly different conditions before. [Nani 2002].

$$\begin{cases} \dot{x}_1 = (a_1 - b_1 x_1) x_1 - k_{12} x_1 x_2 - k_{13} x_1 x_3 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2 \\ \dot{x}_2 = S_2 - k_{21} x_2 x_1 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 \\ \dot{x}_3 = r_{23} x_2 - k_{31} x_3 x_1 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2 \end{cases}$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases}$$

3.0 The Analysis of Rest Points

In this chapter the generalized models will be analyzed. In particular the rest points or the equilibrium points or fixed points will be computed. Initially the definition and discussion of the terms rest point, hyperbolicity, linearization will be done.

Let $\dot{x} = F(x)$ where $x \in R^n = [x_1, \dots, x_n]^T$ be a system of ordinary differential equations.

We call x a rest point of the system if $\dot{x} = F(x_0) = 0$

To compute the rest point of the system, we set $\dot{x} = F(x) = 0$ and solve for all possible values which make this possible.

An equilibrium point of a dynamical system generated by a system of ODE is a solution that does not change with time.

3.0.1 Definition Hyperbolicity of Rest Points

Consider $\dot{x} = F(x)$ where $x \in R^n = [x_1, \dots, x_n]^T$

$F_i \in C'(R^n, R) \forall i = \{1, \dots, n\}$, where C' stands for continuous and at least once *differentiable*

The rest point $x_0 \in R^n$ is hyperbolic if :

- (i) $F(x_0) = 0$ for each $i = \{1, \dots, n\}$ and $x_0 \in R^n$
- (ii) The Jacobian Matrix $[DF(x_0)]$ has no eigen values on the imaginary axis. That is all the eigen values of the Jacobian Matrix have nonzero real parts.

3.0.2 The Jacobian Matrix (of linearization)

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= F_1(x_1, x_2, \dots, x_n) \\ \dot{x}_2 &= F_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= F_n(x_1, x_2, \dots, x_n)\end{aligned}$$

Let $x_0 = [x_{10}, x_{20}, x_{30}, \dots, x_{n0}]$ be a hyperbolic rest point of the system. Then by the Taylor expansions for functions of several variables,

$$\begin{aligned}F_1(x_1, \dots, x_n) &= F_1(x_{10}, \dots, x_{n0}) + \frac{F_{1x_1}(x_{10}, \dots, x_{n0})}{1!}(x_1 - x_{10}) + \frac{F_{1x_2}(x_{10}, \dots, x_{n0})}{1!}(x_2 - x_{20}) + \\ &\quad + \dots + \frac{F_{1x_n}(x_{10}, \dots, x_{n0})}{1!}(x_n - x_{n0}) + NLT \\ F_2(x_1, \dots, x_n) &= F_2(x_{10}, \dots, x_{n0}) + \frac{F_{2x_1}(x_{10}, \dots, x_{n0})}{1!}(x_1 - x_{10}) + \frac{F_{2x_2}(x_{10}, \dots, x_{n0})}{1!}(x_2 - x_{20}) + \\ &\quad + \dots + \frac{F_{2x_n}(x_{10}, \dots, x_{n0})}{1!}(x_n - x_{n0}) + NLT \\ &\vdots \\ F_n(x_1, \dots, x_n) &= F_n(x_{10}, \dots, x_{n0}) + \frac{F_{nx_1}(x_{10}, \dots, x_{n0})}{1!}(x_1 - x_{10}) + \frac{F_{nx_2}(x_{10}, \dots, x_{n0})}{1!}(x_2 - x_{20}) + \\ &\quad + \dots + \frac{F_{nx_n}(x_{10}, \dots, x_{n0})}{1!}(x_n - x_{n0}) + NLT\end{aligned}$$

where NLT stands for Non - Linear Terms

Let

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} x_1 - x_{10} \\ x_2 - x_{20} \\ \vdots \\ x_n - x_{n0} \end{bmatrix} \quad \text{and} \quad \dot{\xi} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_n \end{bmatrix}$$

Then,

$$\dot{\xi}_1 = F_{1x_1}(x_{10}, \dots, x_{n0})\xi_1 + F_{1x_2}(x_{10}, \dots, x_{n0})\xi_2 + \dots + F_{1x_n}(x_{10}, \dots, x_{n0})\xi_n + NLT$$

$$\dot{\xi}_2 = F_{2x_1}(x_{10}, \dots, x_{n0})\xi_1 + F_{2x_2}(x_{10}, \dots, x_{n0})\xi_2 + \dots + F_{2x_n}(x_{10}, \dots, x_{n0})\xi_n + NLT$$

\vdots

$$\dot{\xi}_n = F_{nx_1}(x_{10}, \dots, x_{n0})\xi_1 + F_{nx_2}(x_{10}, \dots, x_{n0})\xi_2 + \dots + F_{nx_n}(x_{10}, \dots, x_{n0})\xi_n + NLT$$

Thus the system can be written as,

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_n \end{bmatrix} = \begin{bmatrix} F_{1x_1}(x_{10}, \dots, x_{n0}) & F_{1x_2}(x_{10}, \dots, x_{n0}) & \dots & F_{1x_n}(x_{10}, \dots, x_{n0}) \\ F_{2x_1}(x_{10}, \dots, x_{n0}) & F_{2x_2}(x_{10}, \dots, x_{n0}) & \dots & F_{2x_n}(x_{10}, \dots, x_{n0}) \\ \vdots & \vdots & & F_{2x_n}(x_{10}, \dots, x_{n0}) \\ F_{nx_1}(x_{10}, \dots, x_{n0}) & F_{nx_2}(x_{10}, \dots, x_{n0}) & \dots & F_{nx_n}(x_{10}, \dots, x_{n0}) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$$

Denote $\begin{bmatrix} F_{1x_1}(x_{10}, \dots, x_{n0}) & F_{1x_2}(x_{10}, \dots, x_{n0}) & \dots & F_{1x_n}(x_{10}, \dots, x_{n0}) \\ F_{2x_1}(x_{10}, \dots, x_{n0}) & F_{2x_2}(x_{10}, \dots, x_{n0}) & \dots & F_{2x_n}(x_{10}, \dots, x_{n0}) \\ \vdots & \vdots & & F_{2x_n}(x_{10}, \dots, x_{n0}) \\ F_{nx_1}(x_{10}, \dots, x_{n0}) & F_{nx_2}(x_{10}, \dots, x_{n0}) & \dots & F_{nx_n}(x_{10}, \dots, x_{n0}) \end{bmatrix}$ by $[DF(x_{10}, \dots, x_{n0})]$

So the system becomes

$$\dot{\xi} = [DF(x_{10}, x_{20}, \dots, x_{n0})]\xi, \quad \xi \in R^n$$

Thus,

$\dot{x} = F(x) \Leftrightarrow \dot{\xi} = [DF(x_{10}, x_{20}, \dots, x_{n0})] \xi$ in the neighborhood of the hyperbolic rest point x_0 ,

where $[DF(x_{10}, \dots, x_{n0})]$ is known as the *Jacobian Matrix of Linearization*.

The Jacobian matrix is also denoted by $J_{[x_{10}, x_{20}, \dots, x_{n0}]}$ and is equal to:

$$J_{[x_{10}, x_{20}, \dots, x_{n0}]} = [DF(x_{10}, \dots, x_{n0})] = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

The Jacobian Matrix represents the best linear approximation to the differentiable functions F_1, F_2, \dots, F_n near the rest point x_0 of the system, and it is also known as the Jacobian Matrix of linearization.

3.0.3 Hartman and Grobman Theorem

Consider the nonlinear system:

$$\begin{cases} \dot{x} = F(x) \\ x(t_0) = c \end{cases} \quad (1)$$

Let $f \in C^1(R^n, R^n)$ be a diffeomorphism. Assume $x_0 \in R^n$ is a hyperbolic rest point of F , such that:

$$\dot{x} = F(x) \quad (1)$$

Let ϕ be the local flow generated by the non-linear system (1)

Let $A = DF(x_0) \in M_{n \times n}(R)$

Then, there exists neighborhoods U and V , and a homeomorphism $h, h: U \rightarrow V$, such that

$$\phi(t, h(x)) = h(x_0 + e^{tA}(x - x_0))$$

whenever

$$x, x_0 + e^{tA}(x - x_0) \in U \quad \text{and} \quad h(x), h(x_0 + e^{tA}(x - x_0)) \in V$$

In particular in a small neighborhood of x_0 , the flow generated by the NLS of ODE is qualitatively similar or isochronally flow equivalent to the flow generated by the linearized system, where $x \in R^n, c \in R^n$, and $F \in C^1(R^n, R) \forall i = \{1, 2, \dots, n\}$, and the LS:

$$\begin{cases} \dot{\xi} = [DF(x_0)]\xi \\ \xi(t_0) = c \end{cases} \quad (2)$$

with $\xi = (x - x_0) \in R^n$, and $[DF(x_0)] \in M_{n \times n}(R)$

Suppose x_0 is a hyperbolic rest point.

Then the flow generated by the NLS (1) is C^0 conjugate to the flow generated by the LS (2) in the neighborhood of the hyperbolic rest point $x_0 \in R^n$ [Amann, 1990].

3.0.4 Linearized Stability for Hyperbolic Equilibria

Let $F \in C^1(R^n, R^n)$ be a diffeomorphism such that $\dot{x} = F(x)$ (1)

generates a continuous dynamical system Φ in $\Omega \subseteq R^n$.

Suppose that $x_0 \in \Omega$ is a hyperbolic rest point of Φ .

Let $\xi := x - x_0 \in R^n$

$$A := DF(x_0) \in M_{n \times n}(R)$$

Then,

$$\dot{\xi} = F(\xi + x_0) = F(x_0) + DF(x_0)\xi + R(\xi)$$

$$\text{where } \lim_{\xi \rightarrow 0} \left[\frac{|R(\xi)|}{|\xi|} \right] \rightarrow 0 \text{ as } \xi \rightarrow 0$$

$$\text{and } F(x_0) = 0$$

Thus $\dot{\xi} = A\xi$ (2) in the neighborhood of x_0 .

In particular (2) is a linearization of (1)

3.0.5 Principles of Linearized Stability

If $x_0 \in R^n$ is a hyperbolic rest point of $\dot{x} = F(x) \dots(1)$, then x_0 is either unstable or asymptotically stable.

In the following sub sections, the basic principles of linearized stability will be discussed in terms of propositions, theorems and conjectures [Amman 1990].

Proposition 3.0.5.1 Asymptotic Stability

If x_0 is a rest point of $\dot{x} = F(x)$ (1) and all the eigenvalues of $A := DF(x_0)$ have negative real part, then x_0 is locally asymptotically stable and,

$$\|e^{tA}x\| \leq e^{-\alpha t}\|x\|, \quad \alpha > 0$$

where $\|\bullet\|$ is the Hilbert norm on R^n . In particular, the origin of the linearized system is a sink.

Proposition 3.0.5.2 Instability

If x_0 is a hyperbolic rest point of $\dot{x} = F(x) \dots(1)$ and all the eigenvalues of $A := DF(x_0)$ have positive real parts, then x_0 is unstable and $\|e^{tA}x\| \leq e^{-\alpha t}\|x\|$, where

$\alpha > 0, \forall t \geq 0, x \in R^n$. In particular, the origin of the linearized system is a source.

Also If x_0 is a hyperbolic rest point of $\dot{x} = F(x) \dots(1)$ and at least one of the eigenvalues has positive real part and at least one has negative real part, then x_0 is unstable.

Hence a hyperbolic rest point is unstable if at least one eigenvalue of the Jacobian Matrix

has positive real part.

3.0.6 Classification of Hyperbolic Rest Points.

From the previous propositions it can be concluded that the stability of equilibrium of a hyperbolic rest point is determined by the sign of the real parts of the eigenvalues of the Jacobian matrix. Depending on the signs of the real part of eigenvalues, a hyperbolic rest point can be classified into sinks, sources and hyperbolic saddles [Alligood et al. 1996].

Let $x_0 \in R^n$ be the rest point of the ODE:

$$\dot{x} = F(x), \quad R^n \rightarrow R^n,$$

Thus, $F(x_0) = 0$

Let the eigen-spectrum corresponding to the linearization of the ODE around x_0 be defined as:

$$\sigma(x_0) = \{\lambda_i \mid \det[\lambda I - [DF(x_0)]] = 0 \quad i = \{1, 2, \dots, k\}\}$$

Then

- (i) The rest point x_0 is called a hyperbolic sink or an attractor if $\text{Re } \lambda_i < 0 \quad \forall i$

In particular a sink is locally asymptotically stable.

- (ii) The rest point x_0 is called a hyperbolic source or a repellor if $\text{Re } \lambda_i > 0 \quad \forall i$

In particular a sink is unstable.

- (iii) The rest point x_0 is called a hyperbolic saddle if $\sigma(x_0) = \sigma_u \oplus \sigma_s$

where σ_u is the unstable manifold consisting of eigenvalues with positive real parts, and σ_s is the stable manifold

Theorem 3.0.6a

If a hyperbolic rest point is a sink, then it is stable.

Proof:

Let $\dot{x} = F(x)$ be an ODE and suppose that x_0 is a rest point of the system, such that x_0 is a sink. Then by definition of sink all eigenvalues of the Jacobian Matrix have negative real parts. Hence by definitions in 3.0.6, x_0 is locally asymptotically stable.

Theorem 3.0.6b

If a hyperbolic rest point is a source, then it is unstable.

Proof:

Again, let $\dot{x} = F(x)$ be an ODE and suppose that x_0 is a rest point of the system, such that x_0 is a source. Then by definition of source all eigenvalues of the Jacobian Matrix have positive real parts. Hence by definitions in 3.0.6, x_0 is unstable.

Theorem 3.0.6c

If a hyperbolic rest point is a saddle, then it is unstable.

Proof:

Once more time, let $\dot{x} = F(x)$ be an ODE and suppose that x_0 is a rest point of the system, such that x_0 is a saddle. Then by definition of saddle at least one eigenvalue of the Jacobian Matrix has positive real part and at least one eigenvalue of the Jacobian Matrix has negative real part. Hence definitions in 3.0.6, x_0 is unstable.

4.0 ANALYSIS OF MODELS

In this section we proceed to study in detail the surge supply model and the logistic source model. Both models are analyzed in detail in R^2 . Later on in the next chapter the models will be analyzed in R^3 .

4.1 The Surge Model

The equations of the surge model have been independently constructed and analyzed under a different setting using simulation techniques by Professor Nani, [Nani 2002]. The surge model represents the mathematical model of military combat. in which the following features are incorporated:

[i] The surge supply rate is defined by the surge function:

$$S = f(x) = axe^{-bx}$$

[ii] There are three combating armies: army #1, army #2 and army #3

[iii] The army #1 is an **expeditionary force** invading a domain D where two two insurgent armies (army #2 and army #3) reside

[iv] In particular army #1 fights army #2 and army #3 in a **three –way- fight**

[v] Army #2 consists of the indigenous people, but the insurgent army #3 is recruited from army #2.

[vi] Army #1 also recruits from army #2

An illustration of the combat scenario between the three armies is shown in figure 1.1

The surge model equations are:

$$\begin{aligned}\dot{x}_1 &= a_1 x_1 e^{-b_1 x_1} - k_{12} x_1 x_2 - k_{13} x_1 x_3 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2 \\ \dot{x}_2 &= S_2 - k_{21} x_2 x_1 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 \\ \dot{x}_3 &= r_{23} x_2 - k_{31} x_3 x_1 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2\end{aligned}\quad \mathbf{S\ 1.1\ a}$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases} \quad \mathbf{S\ 1.1\ b}$$

4.1.2 Nonnegativity, Dissipativity

In this subsection the non negativity and dissipativity of the system is analyzed. The analysis is done for the planes x_1 - x_2 , x_2 - x_3 , x_1 - x_3 .

4.1.2.1 Analysis of the model in the $x_1 - x_2$ Plane

In the plane x_1 - x_2 , the S 1.1a reduces to the following:

$$\dot{x}_1 = a_1 x_1 e^{-b_1 x_1} - k_{12} x_1 x_2 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2$$

$$\dot{x}_2 = S_2 - k_{21} x_2 x_1 - r_{23} x_2 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2$$

Consider the logistic supply $S_2 = (a_2 - b_2 x_2) x_2 = a_2 x_2 - b_2 x_2^2$ and let $A_1 = \text{Max}(a_1 e^{b_1 x_1})$

and $M_2 = \text{Max} x_2(t)$, for $t_0 \leq t \leq T$, then:

$$\dot{x}_1 \leq A_1 x_1 - k_1 x_1 + r_{21} M_2 = (A_1 - k_1) x_1 + r_{21} M_2$$

$$\text{Allow } \dot{u} = (A_1 - k_1) u + r_{21} M_2$$

$$\Rightarrow \dot{u} + (k_1 - A_1) u = r_{21} M_2$$

$$u = \frac{r_{21} M_2}{k_1 - A_1} + c e^{-(k_1 - A_1)t}$$

$$\text{So } x_1 \leq \frac{r_{21} M_2}{k_1 - A_1} + c e^{-(k_1 - A_1)t} \rightarrow \frac{r_{21} M_2}{k_1 - A_1} \text{ as } t \rightarrow \infty, \text{ provided that } k_1 \geq A_1$$

$$\Rightarrow 0 \leq x_1 \leq \text{Max} \left[x_{10}, \frac{r_{21} M_2}{k_1 - A_1} \right]$$

$$\text{Similarly } \dot{x}_2 \leq (a_2 - r_{23} - k_2 - r_{21}) x_2 + (-b_2 - k_{22}) x_2^2$$

$$\text{Allow } \dot{u} = (a_2 - r_{23} - k_2 - r_{21}) u + (-b_2 - k_{22}) u^2$$

$$\Rightarrow u + (r_{23} + k_2 + r_{21} - a_2) u = (-b_2 - k_{22}) u^2$$

$$u = \frac{1}{\frac{b_2 + k_{22}}{a_2 - r_{23} - k_2 - r_{21}} + c e^{(r_{23} + k_2 + r_{21} - a_2)t}}$$

$$\text{So } x_2 \leq \frac{1}{\frac{b_2 + k_{22}}{a_2 - r_{23} - k_2 - r_{21}} + c e^{(r_{23} + k_2 + r_{21} - a_2)t}} \rightarrow \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \text{ as } t \rightarrow \infty$$

(provided that $r_{23} + k_2 + r_{21} \leq a_2$)

$$\Rightarrow 0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right]$$

Thus,

$$B_{x_1, x_2} = \left[0 \leq x_1 \leq \text{Max} \left[x_{10}, \frac{r_{21} M_2}{k_1 - A_1} \right], 0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right] \right]$$

is an invariant box.

This implies that any solution which originates in the positive octant

$R_+^n = \{x_i \mid x_i \geq 0 \ i = 1, 2\}$ will remain non-negative and is ultimately bounded. In

particular the system is dissipative such that:

$$\limsup x_i(t) \leq M, \text{ where } M = \max \left[x_{10}, x_{20}, \left[\frac{r_{21} M_2}{k_1 - A_1} \right], \left[\frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right] \right]$$

4.1.2.2 Analysis of the model in the $x_1 - x_3$ Plane

Consider the positive octant $R_+^n = \{x_i \mid x_i \geq 0 \ i = 1, 3\}$

Theorem 4.1.2a

The set:

$$B_{x_1, x_3} = \left[0 \leq x_1 \leq \text{Max} \left[x_{10}, \frac{A_1 - k_1}{k_{11}} \right], 0 \leq x_3 \leq \text{Max} [x_{30}, 0] \right] \text{ is positively invariant with}$$

respect to the solution curves of S1.1a and S1.1.b.

Proof:

$$\dot{x}_1 = a_1 x_1 e^{-b_1 x_1} - k_{13} x_1 x_3 - k_{13} x_1 x_3 - k_1 x_1 - k_{11} x_1^2$$

$$\dot{x}_3 = -k_{31} x_3 x_1 - k_3 x_3 - k_{33} x_3^2$$

$$\text{Then } \dot{x}_1 \leq a_1 x_1 e^{-b_1 x_1} - k_1 x_1 - k_{11} x_1^2$$

$$\text{Let } A_1 = a_1 \text{Max}(e^{-b_1 x_1})$$

$$\Rightarrow x_1 \leq (A_1 - k_1) x_1 - k_{11} x_1^2$$

$$\text{Allow } \dot{u} = (A_1 - k_1) x_1 - k_{11} x_1^2$$

$$\Rightarrow \dot{u} - (A_1 - k_1) x_1 = -k_{11} x_1^2$$

$$u = \frac{1}{\frac{k_{11}}{A_1 - k_1} + c e^{-(A_1 - k_1)t}}$$

$$\text{So } x_1 \leq \frac{1}{\frac{k_{11}}{A_1 - k_1} + c e^{-(A_1 - k_1)t}} \rightarrow \frac{A_1 - k_1}{k_{11}} \text{ as } t \rightarrow \infty, \text{ provided that } k_1 < A_1$$

$$\Rightarrow 0 \leq x_1 \leq \text{Max}\left[x_{10}, \frac{A_1 - k_1}{k_{11}}\right]$$

$$\text{Similarly } \dot{x}_3 \leq -k_3 x_3 - k_{33} x_3^2$$

$$\text{Allow } \dot{u} = -k_3 u - k_{33} u^2$$

$$\Rightarrow \dot{u} + k_3 u = -k_{33} u^2$$

$$u = \frac{1}{-\frac{k_{33}}{k_3} + c e^{k_3 t}}$$

$$\text{So } x_3 \leq \frac{1}{-\frac{k_{33}}{k_3} + c e^{k_3 t}} \rightarrow 0 \text{ as } t \rightarrow \infty$$

$$\Rightarrow 0 \leq x_3 \leq \text{Max}[x_{30}, 0]$$

This implies that any solution which originates in the positive octant

$R_+^n = \{x_i \mid x_i \geq 0 \ i = 1, 2, 3\}$ will remain non-negative and is ultimately bounded. In

particular the system is dissipative such that;

$$\limsup x_i(t) \leq M, \text{ where } M = \max \left[x_{10}, \quad x_{30}, \quad \left[\frac{A_1 - k_1}{k_{11}} \right] \right]$$

4.1.2.3 Analysis of the model in the $x_2 - x_3$ Plane

Consider the positive octant $R_+^n = \{x_i \mid x_i \geq 0 \ i = 2, 3\}$.

Theorem 4.1.2b

The set

$$B_{x_2, x_3} = \left[0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right], 0 \leq x_3 \leq \text{Max} \left[x_{30}, \frac{r_{23} A_2}{k_{32} A_2 + k_3} \right] \right] \text{ is}$$

positively invariant with respect to the solution curves of S1.1a and S1.1.b.

Proof:

$$\dot{x}_2 = a_2 x_2 - b_2 x_2^2 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2$$

$$\dot{x}_3 = r_{23} x_2 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2$$

$$\Rightarrow \dot{x}_2 \leq (a_2 - r_{23} - k_2 - r_{21}) x_2 - (b_2 + k_{22}) x_2^2$$

$$\text{Allow } \dot{u} = (a_2 - r_{23} - k_2 - r_{21}) u - (b_2 + k_{22}) u^2$$

$$\Rightarrow \dot{u} - (a_2 - r_{23} - k_2 - r_{21}) u = -(b_2 + k_{22}) u^2$$

$$u = \frac{1}{\frac{b_2 + k_{22}}{a_2 - r_{23} - k_2 - r_{21}} + c e^{-(a_2 - r_{23} - k_2 - r_{21})t}}$$

$$\text{So } x_2 \leq \frac{1}{\frac{b_2 + k_{22}}{a_2 - r_{23} - k_2 - r_{21}} + ce^{-(a_2 - r_{23} - k_2 - r_{21})t}} \rightarrow \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \text{ as } t \rightarrow \infty$$

(provided that $a_2 - r_{23} - k_2 - r_{21} > 0$)

$$\Rightarrow 0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right]$$

Let $A_2 = \text{Max}(x_2)$ for t such that $t_0 \leq t \leq T$

$$\Rightarrow x_3 \leq r_{23}A_2 - k_{32}x_3x_2 - k_3x_3$$

$$\text{Allow } \dot{u} = r_{23}A_2 - (k_{32}A_2 + k_3)u$$

$$\Rightarrow \dot{u} + (k_{32}A_2 + k_3)u = r_{23}A_2$$

$$u = \frac{r_{23}A_2}{k_{32}A_2 + k_3} + ce^{-(k_{32}A_2 + k_3)t}$$

$$\text{So } x_3 \leq \frac{r_{23}A_2}{k_{32}A_2 + k_3} + ce^{-(k_{32}A_2 + k_3)t} \rightarrow \frac{r_{23}A_2}{k_{32}A_2 + k_3} \text{ as } t \rightarrow \infty$$

$$\Rightarrow 0 \leq x_3 \leq \text{Max} \left[x_{30}, \frac{r_{23}A_2}{k_{32}A_2 + k_3} \right]$$

Thus,

$$B_{x_2, x_3} = \left[0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right], 0 \leq x_3 \leq \text{Max} \left[x_{30}, \frac{r_{23}A_2}{k_{32}A_2 + k_3} \right] \right]$$

This implies that any solution which originates in the positive octant

$R_+^n = \{x_i \mid x_i \geq 0 \ i = 2, 3\}$ will remain non-negative and is ultimately bounded.

In particular the system is dissipative such that,

$$\limsup x_i(t) \leq M, \text{ where } M = \max \left[x_{20}, x_{30}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}}, \frac{r_{23}A_2}{k_{32}A_2 + k_3} \right].$$

4.1.3 Analysis of the Rest Points

In this subsection the rest points of the model will be calculated, analyzed and militarily interpreted. The analysis is done for the planes x_1-x_2 , x_2-x_3 , and x_1-x_3 .

4.1.3.1 Rest Point Analysis in the $x_1 - x_2$ Plane

The rest points are computed by setting $\dot{x}_i(t) = 0$ for $i = 1, 2$. Thus the surge model equations reduce to:

$$\begin{aligned} \dot{x}_1 &= a_1 x_1 e^{-b_1 x_{11}} - k_{12} x_1 x_2 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2 = 0 \\ \dot{x}_2 &= (a_2 - b_2 x_2) x_2 - K_{21} x_2 x_1 - r_{23} x_2 - k_2 x_2 - K_{22} x_2^2 - r_{21} x_2 = 0 \end{aligned}$$

Clearly $E_1 = [0, 0]$ is a rest point of the system.

The second rest point E_2 is obtained as follow :

Let $x_2 = 0$:

$$\begin{aligned} \Rightarrow a_1 x_1 e^{-b_1 x_{11}} - k_1 x_1 &= 0 \\ (a_1 e^{-b_1 x_{11}} - k_1) x_1 &= 0 \end{aligned}$$

Thus,

$$a_1 e^{-b_1 x_{11}} - k_1 = 0$$

$$e^{-b_1 x_{11}} = \frac{k_1}{a_1}$$

$$-b_1 x_1 = \ln\left(\frac{k_1}{a_1}\right)$$

$$x_1 = \frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right)$$

$$\Rightarrow E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$$

Thus in the $X_1 - X_2$ plane the rest points are :

$$E_1 = [0,0] \text{ and } E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$$

4.1.3.2 The Jacobian Matrix of Linearization in the $x_1 - x_2$ Plane

$$\dot{x}_1 = a_1 x_1 e^{-b_1 x_{11}} - k_{12} x_1 x_2 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2 = F_1$$

$$\dot{x}_2 = (a_2 - b_2 x_2) x_2 - k_{21} x_2 x_1 - r_{23} x_2 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 = F_2$$

$$J_{[x_1, x_2]} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}$$

$$J_{[x_1, x_2]} = \begin{bmatrix} a_1 e^{-b_1 x_1} - a_1 b_1 x_1 e^{-b_1 x_1} - k_{12} x_2 - k_1 - 2k_{11} x_1 & -k_{12} x_1 + r_{21} \\ -k_{21} x_2 & a_2 - 2b_2 x_2 - k_{21} x_1 - r_{23} - k_2 - 2k_{22} x_2 - r_{21} \end{bmatrix}$$

$$\text{Then } J_{[0,0]} = \begin{bmatrix} a_1 - k_1 & r_{21} \\ 0 & a_2 - r_{23} - k_2 - r_{21} \end{bmatrix}$$

Hence the eigenvalues are:

$$\lambda_1 = a_1 - k_1 \quad \text{and} \quad \lambda_2 = a_2 - r_{23} - k_2 - r_{21}$$

Similarly,

$$J_{\left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0\right]} = \begin{bmatrix} b_1 k_1 \ln\left(\frac{k_1}{a_1}\right) & \frac{k_{12}}{b_1} \ln\left(\frac{k_1}{a_1}\right) + r_{21} \\ 0 & \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) + a_2 - r_{23} - k_2 - r_{21} \end{bmatrix}$$

Thus the eigenvalues are:

$$\lambda_2 = b_1 k_1 \ln\left(\frac{k_1}{a_1}\right) \quad \text{and} \quad \lambda_3 = \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) + a_2 - r_{23} - k_2 - r_{21}$$

Theorem 4.1.3a

In the x_1 - x_2 plane the system S.1.1a - S.1.1b is such that the rest point $E_1 = [0,0]$ is a

hyperbolic sink and hence an attractor if:

$$a_1 - k_1 < 0 \quad \text{and} \quad a_2 - r_{23} - k_2 - r_{21} < 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.1.3b

In the x_1 - x_2 plane the system S.1.1a-S.1.1b is such that the rest point $E_1 = [0,0]$ is a source and hence a repeller if:

$$a_1 - k_1 > 0 \text{ and } a_2 - r_{23} - k_2 - r_{21} > 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.1.3c

In the x_1 - x_2 plane the system S.1.1a-S.1.1b is such that the rest point $E_1 = [0,0]$ is a hyperbolic saddle if:

$$a_1 - k_1 > 0 \text{ and } a_2 - r_{23} - k_2 - r_{21} < 0$$

or

$$a_1 - k_1 < 0 \text{ and } a_2 - r_{23} - k_2 - r_{21} > 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comments:

By Proposition 3.0.5.1, in the $x_1 - x_2$ plane the rest point $E_2 = [0,0]$ is locally asymptotically stable if:

$$a_1 - k_1 < 0, \text{ and } a_2 - r_{23} - k_2 - r_{21} < 0.$$

Hence, we can conclude that army #1 should not operate the war under the next configuration:

$$a_1 < k_1 \text{ and } a_2 - r_{23} - k_2 < r_{21}.$$

Under this configuration mutual annihilation will occur.

Theorem 4.1.3d

In the x_1 - x_2 pane the system S.1.1a-S.1.1b is such that the rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$

is a hyperbolic sink and hence an attractor if:

$$k_1 - a_1 < 0 \text{ and } \frac{k_{21}}{b_1} \ln\left(\frac{a_1}{k_1}\right) + a_2 - r_{23} - k_2 - r_{21} < 0.$$

Theorem 4.1.3e

In the x_1 - x_2 pane the system S.1.1a-S.1.1b is such that the rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$

is a source and hence a repeller if:

$$k_1 - a_1 > 0 \text{ and } \frac{k_{21}}{b_1} \ln\left(\frac{a_1}{k_1}\right) + a_2 - r_{23} - k_2 - r_{21} > 0.$$

Theorem 4.1.3f

In the x_1 - x_2 pane the system S.1.1a-S.1.1b is such that the rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$ is

a hyperbolic saddle if:

$$k_1 - a_1 > 0 \text{ and } \frac{k_{21}}{b_1} \ln\left(\frac{a_1}{k_1}\right) + a_2 - r_{23} - k_2 - r_{21} < 0$$

or

$$k_1 - a_1 < 0 \text{ and } \frac{k_{21}}{b_1} \ln\left(\frac{a_1}{k_1}\right) + a_2 - r_{23} - k_2 - r_{21} > 0.$$

Comment:

By Proposition 3.0.5.1, in the $x_1 - x_2$ plane the rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$ is locally

asymptotically stable if:

$$k_1 - a_1 < 0 \text{ and } \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) + a_2 - r_{23} - k_2 - r_{21} < 0.$$

From the stability conditions of the rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$, we can conclude that

the army should operate the war under the next configuration:

$$k_1 - a_1 < 0 \text{ and } \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) + a_2 - r_{23} - k_2 - r_{21} < 0.$$

This configuration will lead to the victory of army #1 over army #2

Military Conclusions

The military meaning of $k_1 - a_1 < 0$ is that the defeat of army #2 by army #1 is conditioned to a higher supply rate of army one than its non-combat death rate. In the practice this implies an increasing of the troop supply rate, which will overextend the

army capacity. To avoid this army # 1 should focus more in the quality of its soldiers rather than its quantity. The greatest Chinese Sun Tzu strategist agrees with this point [Sun Tzu (1910) 2003].

4.1.3.3 Analysis of the model in the $x_1 - x_3$ Plane

$$\dot{x}_1 = a_1 x_1 e^{-b_1 x_{11}} - k_{13} x_1 x_3 - k_1 x_1 - k_{11} x_1^2$$

$$\dot{x}_3 = -k_{31} x_3 x_1 - k_3 x_3 - k_{33} x_3^2 + k_{33} x_3^2$$

Clearly $E_1 = [0,0]$ is a rest point of the system

E_2 :

$$x_3 = 0 \text{ in (2)}$$

$$\Rightarrow a_1 x_1 e^{-b_1 x_{11}} - k_1 x_1 = 0$$

$$(a_1 e^{-b_1 x_{11}} - k_1) x_1 = 0$$

$$\text{Thus } a_1 e^{-b_1 x_{11}} - k_1 = 0$$

$$e^{-b_1 x_{11}} = \frac{k_1}{a_1}$$

So,

$$-b_1 x_1 = \ln\left(\frac{k_1}{a_1}\right)$$

$$x_1 = \frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right)$$

$$\Rightarrow E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$$

4.1.3.4 The Jacobian Matrix of Linearization in the $x_1 - x_3$ Plane

$$\dot{x}_1 = a_1 x_1 e^{-b_1 x_1} - K_{13} x_1 x_3 - k_1 x_1 - K_{11} x_1^2 = F_1$$

$$\dot{x}_3 = -K_{31} x_3 x_1 - k_3 x_3 - k_{33} x_3^2 + k_{33} x_3^2 = F_3$$

$$J_{[x_1, x_3]} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 e^{-b_1 x_1} - a_1 x_1 e^{-b_1 x_1} - k_{13} x_3 - k_1 - 2k_{11} x_1 & -k_{13} x_1 \\ -k_{31} x_3 & -k_{31} x_1 - k_3 - 2k_{33} x_3 \end{bmatrix}$$

$$\text{Then } J_{[0,0]} = \begin{bmatrix} a_1 - k_1 & 0 \\ 0 & -k_3 \end{bmatrix}$$

$$\text{Hence } \lambda_1 = a_1 - k_1 \text{ and } \lambda_2 = -k_3$$

Similarly,

$$J_{\left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0\right]} = \begin{bmatrix} k_1 \ln\left(\frac{k_1}{a_1}\right) + 2 \frac{k_1}{b_1} \ln\left(\frac{k_1}{a_1}\right) & \frac{k_{13}}{b_1} \ln\left(\frac{k_1}{a_1}\right) \\ 0 & \frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3 \end{bmatrix}$$

$$\text{Hence } \lambda_3 = k_1 \ln\left(\frac{k_1}{a_1}\right) + 2 \frac{k_1}{b_1} \ln\left(\frac{k_1}{a_1}\right) \text{ and } \lambda_4 = \frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3$$

Theorem 4.1.3g

In the x_1 - x_3 pane the system S.1.1a-S.1.1b is such that the rest point $E_1 = [0,0]$ is a sink

and hence an attractor if:

$$a_1 - k_1 < 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.1.3h

In the x_1 - x_3 plane the system S.1.1a-S.1.1b is such that the rest point $E_1 = [0,0]$ is a hyperbolic saddle if:

$$a_1 - k_1 > 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comment:

By Proposition 3.0.5.1 in the $x_1 - x_3$ plane the rest point $E_2 = [0,0]$ is locally asymptotically stable if:

$$a_1 - k_1 < 0.$$

Hence, we can conclude that army #1 should not operate the war under the next configuration:

$$a_1 < k_1.$$

Under this configuration mutual annihilation will occur.

Theorem 4.1.3i

In the x_1 - x_3 plane the system S.1.1a-S.1.1b is such that the rest point $E_2 = \left[\frac{1}{b_1} \ln \left(\frac{a_1}{k_1} \right), 0 \right]$

is a sink hence an attractor if:

$$k_1 - a_1 < 0 \quad \text{and} \quad \frac{k_{31}}{b_1} \ln \left(\frac{k_1}{a_1} \right) - k_3 < 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.1.3j

In the x_1 - x_3 plane the system S.1.1a-S.1.1b is such that the rest point $E_2 = \left[\frac{1}{b_1} \ln \left(\frac{a_1}{k_1} \right), 0 \right]$

is a source hence a repeller if:

$$k_1 - a_1 > 0 \quad \text{and} \quad \frac{k_{31}}{b_1} \ln \left(\frac{k_1}{a_1} \right) - k_3 > 0$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.1.3k

In the x_1 - x_3 plane the system S.1.1a-S.1.1b is such that the rest point $E_2 = \left[\frac{1}{b_1} \ln \left(\frac{a_1}{k_1} \right), 0 \right]$

is a hyperbolic saddle if:

$$k_1 - a_1 > 0 \quad \text{and} \quad \frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3 < 0$$

or

$$k_1 - a_1 < 0 \quad \text{and} \quad \frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3 > 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comment:

By proposition 3.0.5.1, in the $x_1 - x_3$ plane the rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$ is locally

asymptotically stable if:

$$a_1 - k_1 < 0 \quad \text{and} \quad \frac{k_{31}}{b_1} \ln\left(\frac{a_1}{k_1}\right) k_1 - k_3 < 0$$

Hence from the stability conditions of the rest point, $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0 \right]$ we can

conclude that the army should operate the war under the next configuration:

$$a_1 - k_1 < 0 \quad \text{and} \quad \frac{k_{31}}{b_1} \ln\left(\frac{a_1}{k_1}\right) k_1 - k_3 < 0$$

This configuration will lead to the victory of army #1 over army #3.

Military Conclusions:

The military interpretation of $a_1 < k_1$ is that for army one to defeat army #3, the supply rate of army #1 needs to be higher than its non-combat death rate. This implies increasing the number of troops faster than normally.

4.2 Logistic Model

In this section we proceed to study in detail the logistic source model. The logistic model represents the mathematical model of military combat in which the following features are incorporated:

[i] The logistic supply rate is defined by the logistic function:

$$S = f(x) = ax - bx^2$$

[ii] There are three combating armies: army #1, army #2 and army #3

[iii] The army #1 is an **expeditionary force** invading a domain D where two two insurgent armies (army #2 and army #3) reside

[iv] In particular army #1 fights army #2 and army #3 in a **three –way- fight**

[v] Army #2 consists of the indigenous people, but the insurgent army #3 is recruited from army #2

[vi] Army #1 also recruits from army #2

$$\dot{x}_1 = (a_1 - b_1 x_1)x_1 - k_{12}x_1x_2 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2$$

$$\dot{x}_2 = S_2 - k_{21}x_2x_1 - r_{23}x_2 - k_{23}x_2x_3 - k_2x_2 - k_{22}x_2^2 - r_{21}x_2 \quad \mathbf{S\ 2.1a}$$

$$\dot{x}_3 = r_{23}x_2 - k_{31}x_3x_1 - k_{32}x_3x_2 - k_3x_3 - k_{33}x_3^2$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases} \quad \mathbf{S\ 2.1b}$$

4.2.1 Nonnegativity, Dissipativity

In this subsection the non negativity and dissipativity of the system is analyzed. The analysis is done for the planes x_1-x_2 , x_2-x_3 , x_1-x_3 .

4.2.1.2 Analysis of the model in the $x_1 - x_2$ Plane

In the plane x_1-x_2 , the **S 2.1a** reduces to the following:

$$\dot{x}_1 = (a_1 - b_1 x_1)x_1 - k_{12}x_1x_2 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2$$

$$\dot{x}_2 = (a_2 - b_2 x_2)x_2 - k_{21}x_2x_1 - r_{23}x_2 - k_2x_2 - k_{22}x_2^2 - r_{21}x_2$$

Now Let $M_2 = \text{Max } x_2(t)$, for $t_0 \leq t \leq T$, then :

$$\dot{x}_1 \leq (a_1 - k_{12}M_2 - k_1)x_1 + r_{21}M_2$$

$$\text{Allow } \dot{u} = (a_1 - k_{12}M_2 - k_1)x_1 + r_{21}M_2$$

$$\Rightarrow \dot{u} + (k_{12}M_2 + k_1 - a_1)x_1 = r_{21}M_2$$

$$u = \frac{r_{21}M_2}{k_{12}M_2 + k_1 - a_1} + ce^{-(k_{12}M_2 + k_1 - a_1)t}$$

$$\text{Thus } x_1 \leq \frac{r_{21}M_2}{k_{12}M_2 + k_1 - a_1} + ce^{-(k_{12}M_2 + k_1 - a_1)t} \rightarrow \frac{r_{21}M_2}{k_{12}M_2 + k_1 - a_1} \text{ as } t \rightarrow \infty$$

So $\limsup \|x_1\| \leq \frac{r_{21}M_2}{k_{12}M_2 + k_1 - a_1}$ for $t \geq T_1$

This implies that x_1 is ultimately bounded.

and $0 \leq x_1 \leq \text{Max} \left[x_{10}, \frac{r_{21}M_2}{k_{12}M_2 + k_1 - a_1} \right]$

Similarly, consider the differential inequality

$$\dot{x}_2 \leq (a_2 - r_{23} - k_2 - r_{21})x_2 + (-b_2 - k_{22})x_2^2$$

In particular if $r_{23} + k_2 + r_{21} \leq a_2$, $\exists T_2$ such that for $t \geq T_2$

x_2 satisfies the inequality :

$$\limsup \|x_2\| \leq \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}}$$

The above inequality implies that the system is dissipative and that x_2

is ultimately bounded

Thus,

$$0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right]$$

In particular

$$B_{x_1, x_2} = \left[0 \leq x_1 \leq \text{Max} \left[x_{10}, \frac{r_{21}M_2}{k_{12}M_2 + k_1 - a_1} \right], 0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right] \right]$$

is an invariant box.

This implies that any solution which originates in the positive octant

$R_+^n = \{x_i \mid x_i \geq 0 \ i = 1, 2\}$ will remain non-negative and is ultimately bounded.

In particular the system is dissipative such that:

$\limsup x_i(t) \leq M$, where

$$M = \max \left[x_{10}, \quad x_{20}, \quad \frac{r_{21}M_2}{k_{12}M_2 + k_1 - a_1}, \quad \left[\frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right] \right].$$

4.2.1.1 Analysis of the model in the $x_1 - x_3$ Plane

In the plane x_1-x_3 , the **S 2.1a** reduces to the following:

$$\dot{x}_1 = (a_1 - b_1x_1)x_1 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2$$

$$\dot{x}_3 = -k_{31}x_3x_1 - k_3x_3 - k_{33}x_3^2$$

$$\text{Let } A_1 = a_1 \text{Max}(e^{-bx_1})$$

$$\Rightarrow \dot{x}_1 \leq (a_1 - k_1)x_1 + (-b_1 - k_{11})x_1^2$$

$$\text{Allow } \dot{u} = (a_1 - k_1)u_1 + (-b_1 - k_{11})u_1^2$$

$$\Rightarrow \dot{u} + (k_1 - a_1)u_1 = (-b_1 - k_{11})u_1^2$$

$$u = \frac{1}{\frac{b_1 + k_{11}}{a_1 - k_1} + ce^{-(a_1 - k_1)t}}$$

$$\text{So } x_1 \leq \frac{1}{\frac{b_1 + k_{11}}{a_1 - k_1} + ce^{-(a_1 - k_1)t}} \rightarrow \frac{a_1 - k_1}{b_1 + k_{11}} \text{ as } t \rightarrow \infty \text{ (provided that } a_1 > k_1)$$

$$\Rightarrow 0 \leq x_1 \leq \text{Max} \left[x_{10}, \frac{a_1 - k_1}{b_1 + k_{11}} \right]$$

Similarly $\dot{x}_3 \leq -k_3 x_3 - k_{33} x_3^2$

Allow $\dot{u} = -k_3 u - k_{33} u^2$

$$\Rightarrow \dot{u} + k_3 u = -k_{33} u^2$$

$$u = \frac{1}{-\frac{k_{33}}{k_3} + ce^{k_3 t}}$$

$$\text{So } x_3 \leq \frac{1}{-\frac{k_{33}}{k_3} + ce^{k_3 t}} \rightarrow 0 \text{ as } t \rightarrow \infty$$

$$\Rightarrow 0 \leq x_3 \leq \text{Max}[x_{30}, 0]$$

Thus,

$$B_{x_1, x_2} = \left[0 \leq x_1 \leq \text{Max}\left[x_{10}, \frac{a_1 - k_1}{b_1 + k_{11}}\right], 0 \leq x_3 \leq \text{Max}[x_{30}, 0] \right]$$

is an invariant box.

This implies that any solution which originates in the positive octant

$R_+^n = \{x_i \mid x_i \geq 0 \ i = 1, 2\}$ will remain non-negative and is ultimately bounded.

In particular the system is dissipative such that;

$$\limsup x_i(t) \leq M, \text{ where } M = \max\left[x_{10}, x_{30}, \frac{a_1 - k_1}{b_1 + k_{11}}, 0\right]$$

4.2.1.1 Analysis of the Model in the $x_2 - x_3$ Plane

In the plane $x_2 - x_3$, the **S 2.1a** reduces to the following:

$$\dot{x}_2 = a_2 x_2 - b_2 x_2^2 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2$$

$$\dot{x}_3 = r_{23} x_2 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2$$

For this case the calculations go exactly as in the surge model.

$$\text{So } x_2 \leq \frac{1}{\frac{b_2 + k_{22}}{a_2 - r_{23} - k_2 - r_{21}} + ce^{-(a_2 - r_{23} - k_2 - r_{21})t}} \rightarrow \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \text{ as } t \rightarrow \infty$$

(provided that $a_2 - r_{23} - k_2 - r_{21} > 0$)

$$\Rightarrow 0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right]$$

$$\text{and } x_3 \leq \frac{r_{23} A_2}{k_{32} A_2 + k_3} + ce^{-(k_{32} A_2 + k_3)t} \rightarrow \frac{r_{23} A_2}{k_{32} A_2 + k_3} \text{ as } t \rightarrow \infty$$

$$\Rightarrow 0 \leq x_3 \leq \text{Max} \left[x_{30}, \frac{r_{23} A_2}{k_{32} A_2 + k_3} \right]$$

Thus,

$$B_{x_2, x_3} = \left[0 \leq x_2 \leq \text{Max} \left[x_{20}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}} \right], 0 \leq x_3 \leq \text{Max} \left[x_{30}, \frac{r_{23} A_2}{k_{32} A_2 + k_3} \right] \right]$$

is an invariant box.

This implies that any solution which originates in the positive octant

$R_+^n = \{x_i \mid x_i \geq 0 \ i = 1, 2\}$ will remain non-negative and is ultimately bounded.

In particular the system is dissipative such that:

$$\limsup x_i(t) \leq M, \text{ where } M = \max \left[x_{20}, x_{30}, \frac{a_2 - r_{23} - k_2 - r_{21}}{b_2 + k_{22}}, \frac{r_{23}A_2}{k_{32}A_2 + k_3} \right]$$

4.2.2 Analysis of the Rest Points

In this subsection the rest points of the model will be calculated, analyzed and militarily interpreted. The analysis is done for the planes x_1 - x_2 , x_1 - x_3 , and x_2 - x_3 .

4.2.2.1 Rest Point Analysis in the $x_1 - x_2$ Plane

The rest points are computed by setting $\dot{x}_i(t) = 0$ for $i = 1, 2$. Thus the model equations reduce to:

$$\dot{x}_1 = (a_1 - b_1x_1)x_1 - k_{12}x_1x_2 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2 \quad (1)$$

$$\dot{x}_2 = (a_2 - b_2x_2)x_2 - k_{21}x_2x_1 - r_{23}x_2 - k_2x_2 - K_{22}x_2^2 - r_{21}x_2 \quad (2)$$

Evidently $E_1 = [0,0]$ is a rest point of the system

$$x_1(a_1 - k_1 - b_1x_1 - k_{11}x_1) = 0$$

$$a_1 - k_1 - b_1x_1 - k_{11}x_1 = 0$$

$$-(b_1 + k_{11})x_1 = k_1 - a_1$$

$$x_1 = \frac{a_1 - k_1}{b_1 + k_{11}}, (a_1 > k_1)$$

$$\Rightarrow E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right], \text{ where } a_1 > k_1.$$

4.2.2.2 The Jacobian Matrix of Linearization in the $x_1 - x_2$ Plane

$$\dot{x}_1 = (a_1 - b_1 x_1)x_1 - k_{12}x_1x_2 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2 = F_1$$

$$\dot{x}_2 = (a_2 - b_2x_2)x_2 - k_{21}x_2x_1 - r_{23}x_2 - k_2x_2 - k_{22}x_2^2 - r_{21}x_2 = F_2$$

$$J_{[x_1, x_2]} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 - 2b_1x_1 - k_{12}x_2 - k_1 - 2k_{11}x_1 & -k_{12}x_1 + r_{21} \\ -k_{21}x_2 & a_2 - 2b_2x_2 - k_{21}x_1 - r_{23} - k_2 - 2k_{22}x_2 - r_{21} \end{bmatrix}$$

$$\text{Then } J_{[0,0]} = \begin{bmatrix} a_1 - k_1 & r_{21} \\ 0 & a_2 - r_{23} - k_2 - r_{21} \end{bmatrix}$$

$$\text{Hence } \lambda_1 = a_1 - k_1 \quad \text{and} \quad \lambda_2 = a_2 - r_{23} - k_2 - r_{21}$$

Similarly,

$$J_{\left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0\right]} = \begin{bmatrix} a_1 - 2b_1 \frac{a_1 - k_1}{b_1 + k_{11}} - k_1 - 2k_{11} \frac{a_1 - k_1}{b_1 + k_{11}} & -k_{12} \frac{a_1 - k_1}{b_1 + k_{11}} + r_{21} \\ 0 & a_2 - k_{12} \frac{a_1 - k_1}{b_1 + k_{11}} - r_{23} - k_2 - r_{21} \end{bmatrix}$$

$$\text{Hence } \lambda_3 = a_1 - \left[k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \right] \quad \text{and} \quad \lambda_4 = a_2 - \left[r_{23} + k_2 + r_{21} + k_{21} \frac{a_1 - k_1}{b_1 + k_{11}} \right].$$

By Theorem 4.1.3a in the $x_1 - x_2$ plane the system S.2.1a - S.2.1b is such that the rest point

$E_1 = [0,0]$ is a sink hence an attractor if:

$$a_1 - k_1 < 0 \quad \text{and} \quad a_2 - r_{23} - k_2 - r_{21} < 0$$

By Theorem 4.1.3b In the $x_1 - x_2$ plane the system S.2.1a - S.2.1b is such that the rest point

$E_1 = [0,0]$ is a source and hence a repeller if:

$$a_1 - k_1 > 0 \text{ and } a_2 - r_{23} - k_2 - r_{21} > 0.$$

By theorem 4.1.3c in the x_1 - x_2 plane the system S.2.1a - S.2.1b is such that the rest point

$E_1 = [0,0]$ is a hyperbolic saddle if:

$$a_1 - k_1 > 0 \text{ and } a_2 - r_{23} - k_2 - r_{21} < 0$$

or

$$a_1 - k_1 < 0 \text{ and } a_2 - r_{23} - k_2 - r_{21} > 0.$$

Comment:

From the stability conditions of the rest point $E_1 = [0,0]$, we can conclude that the army should not operate the war under the next configuration:

$$a_1 < k_1 \text{ and } a_2 - r_{23} - k_2 < r_{21}$$

Under this configuration mutual annihilation will occur.

Theorem 4.2.2a

In the $x_1 - x_2$ plane the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is a sink hence an attractor if:

$$a_1 < k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \text{ and } a_2 < r_{23} + k_2 + r_{21} + k_{21} \frac{a_1 - k_1}{b_1 + k_{11}}.$$

Theorem 4.2.2b

In the $x_1 - x_2$ plane the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is a source hence a repeller if:

$$a_1 > k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad a_2 > r_{23} + k_2 + r_{21} + k_{21} \frac{a_1 - k_1}{b_1 + k_{11}}.$$

Theorem 4.2.2c

In the $x_1 - x_2$ plane the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is a hyperbolic saddle if:

$$a_1 > k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad a_2 < r_{23} + k_2 + r_{21} + k_{21} \frac{a_1 - k_1}{b_1 + k_{11}}$$

or

$$a_1 < k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad a_2 > r_{23} + k_2 + r_{21} + k_{21} \frac{a_1 - k_1}{b_1 + k_{11}}.$$

Comments:

The contrary, the army should operate the war under the configuration:

$$a_1 > k_1, \quad a_1 < k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad a_2 < r_{23} + k_2 + r_{21} + k_{21} \frac{a_1 - k_1}{b_1 + k_{11}}.$$

This configuration will lead to the victory of army one over army two.

4.2.2.3 Rest Point Analysis in the $x_1 - x_3$ Plane

$$\dot{x}_1 = (a_1 - b_1 x_1)x_1 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 = 0 \quad (1)$$

$$\dot{x}_3 = -k_{31}x_3x_1 - k_3x_3 - k_{33}x_3^2 = 0 \quad (2)$$

Clearly $E_1 = [0,0]$ is a rest point of the system.

E_2 :

$$x_3 = 0 \text{ in (2)}$$

$$\Rightarrow (a_1 - b_1 x_1)x_1 - k_1x_1 - k_{11}x_1^2 = 0$$

$$(a_1 - b_1 x_1 - k_1 - k_{11}x_1)x_1 = 0$$

$$a_1 - k_1 - (b_1 + k_{11})x_1 = 0$$

$$x_1 = \frac{a_1 - k_1}{b_1 + k_{11}}$$

$$\Rightarrow E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$$

4.2.2.4 The Jacobian Matrix of Linearization in the $x_1 - x_3$ Plane

$$\dot{x}_1 = (a_1 - b_1 x_1)x_1 - K_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 = F_1$$

$$\dot{x}_3 = -k_{31}x_3x_1 - k_3x_3 - k_{33}x_3^2 = F_3$$

$$J_{[x_1, x_3]} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 - 2b_1x_1 - k_{13}x_3 - k_1 - 2k_{11}x_1 & -k_{13}x_1 \\ -k_{31}x_3 & -k_{31}x_1 - k_3 - 2k_{33}x_3 \end{bmatrix}$$

Then $J_{[0,0]} = \begin{bmatrix} a_1 & 0 \\ 0 & -k_3 \end{bmatrix}$

Hence $\lambda_1 = a_1$ and $\lambda_2 = -k_3$

Similarly,

$$J_{\begin{bmatrix} \frac{a_1 - k_1}{b_1 + k_{11}}, 0 \end{bmatrix}} = \begin{bmatrix} a_1 - 2b_1 \frac{a_1 - k_1}{b_1 + k_{11}} - k_1 - 2k_{11} \frac{a_1 - k_1}{b_1 + k_{11}} & -k_{13} \frac{a_1 - k_1}{b_1 + k_{11}} \\ 0 & -k_{31} \frac{a_1 - k_1}{b_1 + k_{11}} - k_3 \end{bmatrix}$$

$$\Rightarrow \lambda_3 = a_1 - k_1 - 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad \lambda_4 = -k_{31} \frac{a_1 - k_1}{b_1 + k_{11}} - k_3$$

By theorem 4.1.3g in the x_1 - x_3 plane the system S.2.1a-S.2.1b is such that the rest point

$E_1 = [0,0]$ is a sink and hence an attractor if:

$$a_1 - k_1 < 0$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

By theorem 4.1.3h in the x_1 - x_3 plane the system S.2.1a-S.2.1b is such that the rest point

$E_1 = [0,0]$ is a hyperbolic saddle if:

$$a_1 - k_1 > 0$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comment:

By Proposition 3.0.5.1, in the $x_1 - x_3$ plane the rest point $E_2 = [0,0]$ is locally

asymptotically stable if:

$$a_1 - k_1 < 0.$$

Hence, we can conclude that army #1 should not operate the war under the next configuration:

$$a_1 < k_1.$$

Under this configuration mutual annihilation will occur.

Theorem 4.2.2d

In the x_1-x_3 pane the system S.2.1a-S.2.1b is such that the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is a

sink hence an attractor if:

$$a_1 - k_1 - 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} < 0 \text{ and } -k_{31} \frac{a_1 - k_1}{b_1 + k_{11}} - k_3 < 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.2.2e

In the x_1 - x_3 plane the system S.2.1a-S.2.1b is such that the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is

a source hence a repeller if:

$$a_1 - k_1 - 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} > 0 \text{ and } -k_{31} \frac{a_1 - k_1}{b_1 + k_{11}} - k_3 > 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.2.2f

In the x_1 - x_3 plane the system S.2.1a-S.2.1b is such that the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is

a hyperbolic saddle if:

$$a_1 - k_1 - 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} > 0 \text{ and } -k_{31} \frac{a_1 - k_1}{b_1 + k_{11}} - k_3 < 0$$

or

$$a_1 - k_1 - 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} < 0 \text{ and } -k_{31} \frac{a_1 - k_1}{b_1 + k_{11}} - k_3 > 0.$$

By proposition 3.4.1.1, in the $x_1 - x_3$ plane the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is locally

asymptotically stable if:

$$a_1 < k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad k_{31} \frac{k_1 - a_1}{b_1 + k_{11}} < k_3.$$

By definition, in the $x_1 - x_3$ plane the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$ is a hyperbolic saddle

if:

$$a_1 > k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad k_{31} \frac{k_1 - a_1}{b_1 + k_{11}} < k_3$$

or

$$a_1 < k_1 - 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad k_{31} \frac{k_1 - a_1}{b_1 + k_{11}} > k_3.$$

Comment:

From the stability conditions of the rest point $E_2 = \left[\frac{a_1 - k_1}{b_1 + k_{11}}, 0 \right]$, we can conclude that the

army should operate the war under the next configuration:

$$a_1 > k_1, \quad a_1 < k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad k_{31} \frac{k_1 - a_1}{b_1 + k_{11}} < k_3.$$

This configuration will lead to the victory of army one over army two

The army should not operate the war under any the configurations:

$$a_1 > k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad k_{31} \frac{k_1 - a_1}{b_1 + k_{11}} < k_3$$

or

$$a_1 < k_1 - 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad k_{31} \frac{k_1 - a_1}{b_1 + k_{11}} > k_3.$$

or

$$a_1 > k_1 + 2(b_1 + k_{11}) \frac{a_1 - k_1}{b_1 + k_{11}} \quad \text{and} \quad k_{31} \frac{k_1 - a_1}{b_1 + k_{11}} > k_3.$$

4.2.2.5 Rest Point Analysis in the $x_2 - x_3$ Plane

$$x_2 = a_2 x_2 - b_2 x_2^2 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 = 0 \quad (1)$$

$$\dot{x}_3 = r_{23} x_2 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2 = 0 \quad (2)$$

$$E_1 = [0, 0]$$

E_2 :

$$x_2 = 0 \text{ in (1):}$$

$$\Rightarrow -k_3 x_3 - k_{33} x_3^2 = 0$$

$$x_3 (k_3 - k_{33} x_3) = 0$$

$$x_3 = \frac{k_3}{k_{33}}$$

$$\Rightarrow E_2 = \left[0, \frac{k_3}{k_{33}} \right] \text{ is a rest point of the system.}$$

4.2.2.5 The Jacobian Matrix of Linearization in the $x_2 - x_3$ Plane

$$\dot{x}_2 = a_2 x_2 - b_2 x_2^2 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 = F_2$$

$$x_3 = r_{23} x_2 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2 = F_3$$

$$J_{[x_2, x_3]} = \begin{bmatrix} \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} a_2 x_2 - 2b_2 x_2 - r_{23} - k_{23} x_3 - k_2 - 2k_{22} x_2 - r_{21} & -k_{23} x_2 \\ r_{23} - k_{32} x_3 & k_{32} x_2 - k_3 - 2k_{33} x_3 \end{bmatrix}$$

Then,

$$J_{[0,0]} = \begin{bmatrix} a_2 - r_{23} - k_2 - r_{21} & 0 \\ r_{23} & -k_3 \end{bmatrix}$$

Hence $\lambda_1 = a_2 - r_{23} - k_2 - r_{21}$ and $\lambda_2 = -k_3$

Similarly,

$$J_{\left[0, \frac{k_3}{k_{33}}\right]} = \begin{bmatrix} a_2 - r_{23} - \frac{k_{23} k_3}{k_{33}} - k_2 - r_{21} & 0 \\ r_{23} - \frac{k_{32} k_3}{k_{33}} & -3k_3 \end{bmatrix}$$

Hence $\lambda_3 = a_2 - r_{23} - \frac{k_{23} k_3}{k_{33}} - k_2 - r_{21}$ and $\lambda_4 = -3k_3$

By proposition 3.0.5.1 the rest point $E_1 = [0,0]$ is locally asymptotically stable if:

$$a_2 < r_{23} + k_2 + r_{21}$$

By definition, the rest point $E_1 = [0,0]$ is a hyperbolic saddle if:

$$a_2 > r_{23} + k_2 + r_{21}$$

Comment:

Army two should not operate the war under the configuration:

$$a_2 < r_{23} + k_2 + r_{21}$$

This configuration will lead to mutual annihilation of army two and three.

By proposition 3.0.5.1 the rest point $E_2 = \left[0, \frac{k_3}{k_{33}} \right]$ is locally asymptotically stable if:

$$a_2 < r_{23} + \frac{k_{23}k_3}{k_{33}} + k_2 + r_{21}$$

By definition 3.4.2.3 the rest point $E_2 = \left[0, \frac{k_3}{k_{33}} \right]$ is a hyperbolic saddle if:

$$a_2 > r_{23} + \frac{k_{23}k_3}{k_{33}} + k_2 + r_{21}$$

Comment:

By the stability conditions of the rest point $E_2 = \left[0, \frac{k_3}{k_{33}} \right]$, we can conclude that army

three should operate the war under the next configuration:

$$a_2 < r_{23} + \frac{k_{23}k_3}{k_{33}} + k_2 + r_{21}$$

Under this configuration army three will defeat army two.

f4.3 Analysis of the Rest Points for the General System

This section presents an analysis of the general model with participation of three armies in the conflict. This scenario is more realistic and illustrative than the other models. It gives a realistic and detailed perspective of multi armies-conflict. The model can be expanded to a broader number of armies if needed.

4.3.1 Analysis of the Surge Model

Consider the use of the surge source. Then the system is the form:

$$\begin{cases} \dot{x}_1 = a_1 x_1 e^{-b_1 x_1} - k_{12} x_1 x_2 - k_{13} x_1 x_3 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2 \\ \dot{x}_2 = (a_2 - b_2 x_2) x_2 - k_{21} x_2 x_1 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 \\ \dot{x}_3 = r_{23} x_2 - k_{31} x_3 x_1 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2 \end{cases} \quad \mathbf{4.3.1a}$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases} \quad \mathbf{4.3.1b}$$

Theorem 4.3.1a

The rest points of the system 4.3.1a — 4.3.1b are given by:

$$E_1 = [0, 0, 0] \text{ and } E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0, 0 \right].$$

Proof:

$E_1 = [0, 0, 0]$ is clearly a rest point of the system

$x_1 = 0$ and $x_2 = 0$ in the first and second equation of 4.3.1a

$$\Rightarrow -k_3 x_3 - k_{33} x_3^2 = 0$$

$$x_3(-k_3 - k_{33} x_3) = 0$$

$$-k_{33} x_3 = k_3$$

$$x_3 = -\frac{k_3}{k_{33}}$$

$\Rightarrow E_2 = \left[0, 0, -\frac{k_3}{k_{33}} \right]$ is a non-attainable rest point

$x_2 = 0$ and $x_3 = 0$ in the second and third equation:

$$\Rightarrow -a_1 x_1 e^{-b_1 x_1} - k_1 x_1 = 0$$

$$x_1(a_1 e^{-b_1 x_1} - k_1) = 0$$

$$a_1 e^{-b_1 x_1} = k_1$$

$$-b_1 x_1 = \ln\left(\frac{k_1}{a_1}\right)$$

$$x_1 = \frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right)$$

$\Rightarrow E_3 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0, 0 \right]$, where $a_1 > k_1$.

Theorem 4.3.1b

Suppose there is no recruitment of combatants from army # 2 to army #1, that is $r_{21} = 0$,

then the rest point $E_4 = [0, \hat{x}_2, \hat{x}_3]$ is possible iff \hat{x}_2, \hat{x}_3 solve the system:

$$(2k_{33}\hat{x}_3 + k_{32}\hat{x}_2 + k_3)^2 = (k_{32}\hat{x}_2 + k_3)^2 + 4k_{33}r_{23}\hat{x}_2 \quad (\text{i})$$

$$(b_2 + k_{22})\hat{x}_2 + k_{23}\hat{x}_3 = a_2 - k_2 - r_{23} \quad (\text{ii})$$

Proof:

Let $r_{21} = 0$ as stated in the theorem. Then,

$$(a_2 - b_2x_2)x_2 - r_{23}x_2 - k_{23}x_2x_3 - k_2x_2 - k_{22}x_2^2 = 0$$

$$r_{23}x_2 - k_{32}x_3x_2 - k_3x_3 - k_{33}x_3^2 = 0$$

Thus,

$$k_{33}x_3^2 + (k_{32}x_2 + k_3)x_3 - r_{23}x_2 = 0$$

Hence

$$x_3 = (x_{31}, x_{32}) = \frac{-(k_{32}x_2 + k_3) - \sqrt{(k_{32}x_2 + k_3)^2 + 4k_{33}r_{23}x_2}}{2k_{33}}$$

From the second equation of 4.3.1a:

$$x_2[(k_{22} + b_2)x_2 + k_{23}x_3 - (a_2 - k_2 - r_{23})] = 0$$

$$(k_{22} + b_2)x_2 + k_{23}x_3 = a_2 - k_2 - r_{23}$$

$$x_2 = \frac{a_2 - (k_2 + r_{23} + k_{23}x_3)}{k_{22} + b_2}$$

Comment

If this rest point exists then the army # 1 is defeated by the combine mutual fighting with army # 2 and army # 3. In particular the combined forces of army # 2 and army # 3 will totally obliterate army # 1.

To avoid this situation army # 1 will have to engage in the mutual conflict such that the following conditions are NOT satisfied:

$$(2k_{33}\widehat{x}_3 + k_{23}\widehat{x}_2 + k_2)^2 = (k_{23}\widehat{x}_2 + k_2)^2 + 4k_{33}r_{23}\widehat{x}_2$$

$$k_{22}\widehat{x}_2 + k_{23}\widehat{x}_3 = a_2 - k_2 - r_{23}.$$

4.3.1.1 The Jacobian Matrix of Linearization

$$J_{[x_1, x_2, x_3]} = \begin{bmatrix} a_1 e^{-b_1 x_1} - a_1 b_1 x_1 e^{-b_1 x_1} - k_{12} x_2 - k_{13} x_3 - k_1 - 2k_{11} x_1 & -k_{12} x_1 + r_{21} & -k_{13} x_1 \\ -k_{21} x_2 & a_2 - 2b_2 x_2 - k_{21} x_1 - r_{23} - k_{23} x_3 - k_2 - 2k_{22} x_2 - r_{21} & -k_{23} x_2 \\ -k_{31} x_3 & r_{23} - k_{32} x_3 & -k_{31} x_1 - k_{32} x_2 - k_3 - 2k_{33} x_3 \end{bmatrix}$$

$$\Rightarrow J_{[0,0,0]} = \begin{bmatrix} a_1 - k_1 & r_{21} & 0 \\ 0 & a_2 - r_{23} - k_2 - r_{21} & 0 \\ 0 & r_{23} & -k_3 \end{bmatrix}$$

Hence,

$$\lambda_1 = a_1 - k_1, \quad \lambda_2 = a_2 - r_{23} - k_2 - r_{21}, \quad \lambda_3 = 0.$$

Similarly,

$$J_{\left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0, 0\right]} = \begin{bmatrix} -k_1 \ln\left(\frac{a_1}{k_1}\right) - 2\frac{k_{11}}{b_1} \ln\left(\frac{a_1}{k_1}\right) & -\frac{k_{12}}{b_1} \ln\left(\frac{a_1}{k_1}\right) + r_{21} & -\frac{k_{13}}{b_1} \ln\left(\frac{a_1}{k_1}\right) \\ 0 & a_2 - \frac{k_{21}}{b_1} \ln\left(\frac{a_1}{k_1}\right) - r_{23} - k_2 - r_{21} & 0 \\ 0 & r_{23} & -\frac{k_{31}}{b_1} \ln\left(\frac{a_1}{k_1}\right) - k_3 \end{bmatrix}$$

Thus,

$$\lambda_4 = -k_1 \ln\left(\frac{a_1}{k_1}\right) - 2\frac{k_{11}}{b_1} \ln\left(\frac{a_1}{k_1}\right), \quad \lambda_5 = a_2 - \frac{k_{21}}{b_1} \ln\left(\frac{a_1}{k_1}\right) - r_{23} - k_2 - r_{21},$$

$$\lambda_6 = -\frac{k_{31}}{b_1} \ln\left(\frac{a_1}{k_1}\right) - k_3.$$

Theorem 4.3.1c

The rest point $E_1 = [0, 0, 0]$ of the system is a sink hence it is an attractor if:

$$a_1 - k_1 < 0 \quad \text{and} \quad a_2 - r_{23} - k_2 - r_{21} < 0$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Military Tactic Interpretation 4.3.1a

- “The armies will fight to Mutual Assured Destruction” or mutual annihilation if :

$$a_1 < k_1 \text{ and } a_2 < r_{23} + k_2 + r_{21} \bullet$$

- To avoid the above scenario, the expeditionary army #1 has to change at least one of the above criteria. In particular, the army #1 must effect a surge such that

$$a_1 > k_1 \bullet$$

- Thus the surge sustenance rate constant a_1 must be greater than the rate constant k_1 for non-combat loss of troops.
- Another possibility is to minimize troop loss due to non-combat.
- The constant k_1 can be minimized by giving incentive to the army #1, to avoid desertion or protect them from non-combat accidental deaths, or counsel them from suicide. Also It is possible to reduce the number of tours to prevent fatigue-related accidental death or desertion.

Theorem 4.3.1d

The rest point $E_1 = [0, 0, 0]$ of the system is a source hence it is a repeller if:

$$a_1 - k_1 > 0 \text{ and } a_2 - r_{23} - k_2 - r_{21} > 0 \bullet$$

Proof

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.3.1e

The rest point $E_1 = [0, 0, 0]$ of the system is a hyperbolic saddle if:

$$a_1 - k_1 > 0 \quad \text{and} \quad a_2 - r_{23} - k_2 - r_{21} < 0$$

or

$$a_1 - k_1 < 0 \quad \text{and} \quad a_2 - r_{23} - k_2 - r_{21} > 0.$$

Proof

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comments:

From the stability conditions of the rest point $E_1 = [0, 0, 0]$, it can be concluded that the army should not operate the war under the following configuration of parameters:

$$: a_1 - k_1 < 0 \quad \text{and} \quad a_2 - r_{23} - k_2 - r_{21} < 0.$$

Theorem 4.3.1f

The rest point $E_2 = \left[\frac{1}{b_1} \ln \left(\frac{a_1}{k_1} \right), 0, 0 \right]$ of the system is a sink hence it is an attractor if:

$$k_1 < a_1, \quad a_2 + \frac{k_{21}}{b_1} \ln \left(\frac{k_1}{a_1} \right) - (r_{23} + k_2 + r_{21}) < 0, \quad \frac{k_{31}}{b_1} \ln \left(\frac{k_1}{a_1} \right) - k_3 < 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Military Tactic Interpretation 4.3.1f

- If the above criterion holds then army #1 is victorious whereas the other armies are decimated.
- Thus the mathematical criteria $k_1 < a_1$, $a_2 + \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - (r_{23} + k_2 + r_{21}) < 0$,
and $\frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3 < 0$ must be maintained through the entire duration of the conflict.
- Any violation of the above criteria will change the victorious outcome.
- The basic strategy requirement is to ensure that the search has a higher rate of supply. Also that rate constant a_1 is greater than k_1 .

Theorem 4.3.1g

The rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{k_1}{a_1}\right), 0, 0 \right]$ of the system is a source hence it is a repeller if:

$$\left(k_1 + 2 \frac{k_{11}}{b_1} \right) \ln\left(\frac{k_1}{a_1}\right) > 0, \quad a_2 + \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - (r_{23} + k_2) > 0, \quad \frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3 > 0.$$

That is if:

$$k_1 > a_1, \quad a_2 + \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) > (r_{23} + k_2), \quad \frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) > k_3.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comments

- If army # 1 wants to be victorious then it must mobilize its logistics such that the above criteria do not hold.
- In particular army # 2 and army # 3 will **avoid** annihilation if the above criteria hold.

Theorem 4.3.1h

The rest point $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{k_1}{a_1}\right), 0, 0 \right]$ of the system is hyperbolic saddle if at least one of

the eigenvalues; $\lambda_4 = -k_1 \ln\left(\frac{a_1}{k_1}\right) - 2\frac{k_{11}}{b_1} \ln\left(\frac{a_1}{k_1}\right)$, $\lambda_5 = a_2 - \frac{k_{21}}{b_1} \ln\left(\frac{a_1}{k_1}\right) - r_{23} - k_2 - r_{21}$,

$\lambda_6 = -\frac{k_{31}}{b_1} \ln\left(\frac{a_1}{k_1}\right) - k_3$ is positive and at least one is negative.

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comments

If the rest point is a hyperbolic saddle under the above criteria, then the battle outcomes are neither favorable to either of the armies.

4.3.2 The Logistic Model

Consider the use of the logistic source in army #1. Then the system is the form:

$$\begin{cases} \dot{x}_1 = (\alpha_1 - \beta_1 x_1)x_1 - k_{12}x_1x_2 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2 \\ \dot{x}_2 = (\alpha_2 - \beta_2 x_2)x_2 - k_{21}x_2x_1 - r_{23}x_2 - k_{23}x_2x_3 - k_2x_2 - k_{22}x_2^2 - r_{21}x_2 \\ \dot{x}_3 = r_{23}x_2 - k_{31}x_3x_1 - k_{32}x_3x_2 - k_3x_3 - k_{33}x_3^2 \end{cases} \quad 4.3.2a$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases} \quad 4.3.2b$$

Theorem 4.3.2a

The rest points of the system 4.3.1a — 4.3.1b are given by:

$$E_1 = [0, 0, 0]$$

$$E_2 = \left[\frac{1}{\beta_1} \ln \left(\frac{k_1}{\alpha_1} \right), 0, 0 \right].$$

Proof:

$E_1 = [0, 0, 0]$ is clearly a rest point of the system

$x_1 = 0$ and $x_2 = 0$ in the first and second equation of 4.3.1a

$$\Rightarrow -k_3x_3 - k_{33}x_3^2 = 0$$

$$x_3(-k_3 - k_{33}x_3) = 0$$

$$-k_{33}x_3 = k_3$$

$$x_3 = -\frac{k_3}{k_{33}}$$

$$\Rightarrow E_2 = \left[0, 0, -\frac{k_3}{k_{33}} \right] \text{ is a non-attainable rest point}$$

$x_2 = 0$ and $x_3 = 0$ in the second and third equation:

$$\begin{aligned}(\alpha - \beta x_1)x_1 - k_1 x_1 - k_{11} x_1^2 &= 0 \\x_1(\alpha - \beta x_1 - k_1 - k_{11} x_1) &= 0 \\(-\beta - k_{11})x_1 &= k_1 - \alpha \\x_1 &= \frac{\alpha_1 - k_1}{\beta_1 + k_{11}}\end{aligned}$$

$$\Rightarrow E_3 = \left[\frac{\alpha_1 - k_1}{\beta_1 + k_{11}}, 0, 0 \right].$$

Theorem 4.3.2.b

Suppose there is no recruitment of combatants from army # 2 to army #1, that is $r_{21} = 0$,

then the rest point $E_4 = [0, \hat{x}_2, \hat{x}_3]$ is possible iff \hat{x}_2, \hat{x}_3 solve the system:

$$(2k_{33}\hat{x}_3 + k_{32}\hat{x}_2 + k_3)^2 = (k_{32}\hat{x}_2 + k_3)^2 + 4k_{33}r_{23}\hat{x}_2 \quad \text{(i)}$$

$$(k_{22} + b_2)\hat{x}_2 + k_{23}\hat{x}_3 = a_2 - k_2 - r_{23} \quad \text{(ii)}$$

Proof:

The proof follows directly from theorem 4.3.1b

4.3.2.1 The Jacobian Matrix of Linearization

$$J_{[x_1, x_2, x_3]} = \begin{bmatrix} \alpha_{11} - 2\beta_1 x_1 - k_{12} x_2 - k_{13} x_3 - k_1 - 2k_{11} x_1 & -k_{12} x_1 + r_{21} & -k_{13} x_1 \\ -k_{21} x_2 & \alpha_{22} - 2\beta_{22} x_2 - k_{21} x_1 - r_{23} - k_{23} x_3 - k_2 - 2k_{22} x_2 - r_{21} & -k_{23} x_2 \\ -k_{31} x_3 & r_{23} - k_{32} x_3 & -k_{31} x_1 - k_{32} x_2 - k_3 - 2k_{33} x_3 \end{bmatrix}$$

$$\Rightarrow J_{[0,0,0]} = \begin{bmatrix} \alpha - k_1 & r_{21} & 0 \\ 0 & a_2 - r_{23} - k_2 - r_{21} & 0 \\ 0 & r_{23} & -k_3 \end{bmatrix}$$

Hence, the eigen-spectrum for $E_1 = [0, 0, 0]$ is:

$$\sigma(E_1[0,0,0]) = \{ \alpha - k_1, \lambda_2 = a_2 - r_{23} - k_2 - r_{21}, \lambda_3 = -k_3 \}$$

Similarly,

$$J_{\left[\begin{smallmatrix} a_1 - k_1 \\ b_1 + k_1 \end{smallmatrix}, 0, 0 \right]} = \begin{bmatrix} \alpha - k_1 - 2(\alpha - k_1) & -k_{12} \frac{\alpha - k_1}{\beta + k_{11}} + r_{21} & -k_{13} \frac{\alpha - k_1}{\beta + k_{11}} \\ 0 & a_2 - k_{21} \frac{\alpha - k_1}{\beta + k_{11}} - r_{23} - k_2 - r_{21} & 0 \\ 0 & r_{23} & -k_{31} \frac{\alpha - k_1}{\beta + k_{11}} - k_3 \end{bmatrix}$$

Thus the eigen-spectrum of $E_2 = \left[\frac{1}{\beta_1} \ln \left(\frac{k_1}{\alpha_1} \right), 0, 0 \right]$ is:

$$\sigma(E_2 \left(\frac{1}{\beta_1} \ln \left(\frac{k_1}{\alpha_1} \right), 0, 0 \right)) = \{ \lambda_4 = \alpha - k_1 - 2(\alpha - k_1), \lambda_5 = a_2 - k_{21} \frac{\alpha - k_1}{\beta + k_{11}} - r_{23} - k_2 - r_{21}$$

$$\lambda_6 = -k_{31} \frac{\alpha - k_1}{\beta + k_{11}} - k_3 \}$$

Theorem 4.3.2c

The rest point $E_1 = [0, 0, 0]$ of the system is a sink hence it is an attractor if:

$$\alpha_1 - k_1 < 0 \quad \text{and} \quad \alpha_2 - r_{23} - k_2 - r_{21} < 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Military Tactic Interpretation 4.3.2c

- “The armies will fight to Mutual Assured Destruction” or mutual annihilation if :

$$C_1 < \frac{k_1}{\beta_1} \text{ and } \alpha_2 < r_{23} + k_2 + r_{21} , \text{ where } C_1 = \frac{\alpha_1}{\beta_1}$$

In particular C_1 is the logistic maximum for army # 1.

This outcome is analogous to the one observed if the surge rate function is the input function for army # 1, but the rate constants have different interpretations.

- It should be noted that the armies may not have to fight to the bitter end of *zero-end result*, because one of the armies may capitulate before that occur.
- To avoid the above scenario, the expeditionary army #1 has to change at least one of the above criteria. In particular, the army #1 must effect a surge such that:

$$\alpha_1 > k_1.$$

- Thus the logistic rate constant α_1 must be greater than the rate constant k_1 for non-combat loss of troops.
- Another possibility is to minimize troop loss due to non-combat.

The constant k_1 can be minimized by giving incentive to the army #1, to avoid desertion or protect them from non-combat accidental deaths, or counsel them from suicide. Also it is possible to reduce the number of tours to prevent fatigue-related accidental death or desertion.

Theorem 4.3.2d

The rest point $E_1 = [0, 0, 0]$ of the system is a source hence it is a repeller if:

$$\alpha_1 - k_1 > 0 \quad \text{and} \quad \alpha_2 - r_{23} - k_2 - r_{21} > 0.$$

Proof

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comments

The commanders of the three armies obviously will choose battle field conditions such that the above criteria will hold. In particular if each commander would like to destroy the other two armies in order to survive, then this will lead to the rest point being hyperbolic sink with the criteria as stated in theorem 4.3.2a.

On the other hand, if all the three armies fight with caution then this rest point will remain a source.

It should be noted that the initial troop build-up at the start of the combat determines the relative rate at which each army approaches extermination.

Theorem 4.3.2e

The rest point $E_1 = [0, 0, 0]$ of the system is a hyperbolic saddle if:

$$\alpha_1 - k_1 > 0 \quad \text{and} \quad \alpha_2 - r_{23} - k_2 - r_{21} < 0$$

or

$$\alpha_1 - k_1 < 0 \quad \text{and} \quad \alpha_2 - r_{23} - k_2 - r_{21} > 0.$$

Proof

The proof follows directly from the theorem of linearized stability and the stability propositions.

Theorem 4.3.2f

The rest point $E_3 = \left[\frac{\alpha_1 - k_1}{\beta_1 + k_{11}}, 0, 0 \right]$ of the system is a sink hence it is an attractor if:

$$\alpha_1 - k_1 - 2(\alpha_1 - k_1) < 0, \quad \alpha_2 - k_{21} \frac{\alpha_1 - k_1}{\beta_1 + k_1} - r_{23} - k_2 - r_{21} < 0, \quad \text{and} \quad k_{31} \frac{k_1 - \alpha_1}{\beta + k_1} - k_3 < 0.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Comments

The rest point $E_3 = \left[\frac{\alpha_1 - k_1}{\beta_1 + k_{11}}, 0, 0 \right]$ is different from the rest point $E_1 = [0, 0, 0]$

and if $\alpha_1 > k_1$.

A necessary but not sufficient condition for the existence of this rest point is that

$$\alpha_1 > k_1.$$

Military Tactic Interpretation 4.3.2f

- If the above criterion holds then army #1 is victorious whereas the other armies are decimated.

Thus The mathematical criteria:

$$\alpha_1 < k_1, a_2 - k_{21} \frac{\alpha_1 - k_1}{\beta_1 + k_1} < r_{23} + k_2 + r_{21}, \text{ and } k_{31} \frac{k_1 - \alpha_1}{\beta_1 + k_1} < k_3$$

must be maintained through the entire duration of the conflict.

- Any violation of the above criteria will change the victorious outcome. The battle of Aspern gives a good illustration of this failure [Eggenberger 2007].

Theorem 4.3.2g

The rest point $E_3 = \left[\frac{\alpha - k_1}{\beta + k_{11}}, 0, 0 \right]$ of the system is a source hence it is a repeller if:

That is if:

$$\alpha - k_1 > 2(\alpha - k_1), a_2 - k_{21} \frac{\alpha - k_1}{\beta + k_1} > r_{23} + k_2 + r_{21}, \text{ and } k_{31} \frac{k_1 - \alpha}{\beta + k_1} > k_3.$$

Proof:

The proof follows directly from the theorem of linearized stability and the stability propositions.

Military Tactic Interpretation 4.3.2g

If the army # 1 wants to win the conflict, then it has to avoid fighting the battles under the above criteria, which evidently will let victory literally escape from its hands.

Theorem 4.3.2h

The rest point $E[\hat{x}_1, \hat{x}_2, \hat{x}_3]$ exists if the system shows persistence such that,

$$\liminf x_i(t) > \delta_i > 0 \quad i = 1, 2, 3$$

Proof:

The proof follows directly from the definition of persistence and the use of the Butler-McGehee Lemma [Nani 1998].

Comments:

In this situation the long term outcome of the conflict leads to co-existence.

In particular none of the armies can defeat the other conclusively.

5.0 SUMMARY, DISCUSSION, CONCLUSIONS

FUTURE WORK

In this thesis elaborate mathematical models were constructed to depict various scenarios of military combat. The advantage of mathematical models is that, it enables the battle field commanders to have insight of their tactical maneuvers.

Mathematical model serves as another weapon in the arsenal of the armies.

In fighting insurgencies there are a lot of frustrations which results from not using the right strategy from the onset of the counterinsurgencies. One of the critical variables involved is:

[i] Initial troop buildup the initial troop buildup

[ii]The rotation of troops and the supply rate of reinforcements

[iii]The supply rate of troops.

[iv]The timing of the surge in troop numbers

[v]The knowledge of the possible mathematical outcomes of the conflict

5.1 Summary , Discussions, Conclusions of the Generalized Lanchester Models

The Lanchester's linear law (Un-directed Fired Model) has the form:

$$\begin{cases} \dot{x}_1 = S_1 - a_{12}x_1x_2 - a_1x_1 - k_{12}x_2 \\ \dot{x}_2 = S_2 - a_{21}x_1x_2 - a_2x_2 - k_{21}x_1 \\ x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \end{cases}$$

where x_1 = guerrilla army #1

x_2 = guerrilla army #2

a_{12} = The specific inter-army killing rate constant. The number of army # 1 killed per unit time during guerrilla combat engagement with army # 2.

a_{21} = The specific inter-army killing rate constant. The number of army # 2 killed per unit time during guerrilla combat engagement with army # 1.

The outcomes of this model are:

$$E_1 = \left[0, \frac{S_1 - S_2}{k_{12} - a_2} \right]$$

where $k_{12} > a_2$ and $S_1 > S_2$

$$E_2 = \left[\frac{S_2 - S_1}{k_{21} - a_1}, 0 \right]$$

where $k_{21} > a_1$ and $S_2 > S_1$

The third equilibrium point $E_3 = [X_1, X_2]$ exists if the following two surfaces:

$$\Gamma_1 : S_1 = a_{12}x_1x_2 + a_1x_1 + k_{12}x_2$$

$$\Gamma_2 : S_2 = a_{21}x_1x_2 + a_2x_2 + k_{21}x_1$$

intersect at the point $[X_1, X_2]$

$E_3 = [X_1, X_2]$ depicts the interior equilibrium point, with the specifications given in

Table 1.1.

5.2 Summary, Discussions, Conclusions of the Generalized Surge Model

The generalized surge model has the form:

$$\begin{cases} \dot{x}_1 = a_1 x_1 e^{-b_1 x_1} - k_{12} x_1 x_2 - k_{13} x_1 x_3 - k_1 x_1 - k_{11} x_1^2 + r_{21} x_2 \\ \dot{x}_2 = (a_2 - b_2 x_2) x_2 - k_{21} x_2 x_1 - r_{23} x_2 - k_{23} x_2 x_3 - k_2 x_2 - k_{22} x_2^2 - r_{21} x_2 \\ \dot{x}_3 = r_{23} x_2 - k_{31} x_3 x_1 - k_{32} x_3 x_2 - k_3 x_3 - k_{33} x_3^2 \end{cases} \quad \mathbf{4.3.1a}$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases} \quad \mathbf{4.3.1b}$$

In particular the rest points are:

$$E_1 = [0, 0, 0]$$

$$E_2 = \left[\frac{1}{b_1} \ln \left(\frac{k_1}{a_1} \right), 0, 0 \right]$$

Military Tactic Interpretation on $E_1 = [0, 0, 0]$

“The armies will fight to **Mutual Assured Destruction**” or mutual annihilation if :

$$a_1 < k_1 \text{ and } a_2 < r_{23} + k_2 + r_{21}.$$

- To avoid the above scenario, the expeditionary army #1 has to change at least one of the above criteria. In particular, the army #1 must effect a surge such that $a_1 > k_1$.
- Also the surge sustenance rate constant a_1 must be greater than the rate constant k_1 for non-combat loss of troops.
- Another possibility is to minimize troop loss due to non-combat.
- The constant k_1 can be minimized by giving incentive to the army #1, to avoid desertion or protect them from non-combat accidental deaths, or counsel them from suicide. Also It is possible to reduce the number of tours to prevent fatigue-related accidental death or desertion. Another similar approach to reduce casualties is through defensive tactics [Clausewitz (1942) 2003].

Military Tactic Interpretation on $E_2 = \left[\frac{1}{b_1} \ln\left(\frac{a_1}{k_1}\right), 0, 0 \right]$

This rest points occurs when $k_1 < a_1$ and is locally asymptotically stable when:

$$\left(k_1 + 2 \frac{k_{11}}{b_1}\right) \ln\left(\frac{k_1}{a_1}\right) < 0, \quad a_2 + \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - (r_{23} + k_2) < 0, \quad \frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3 < 0$$

- If the above criterion holds then army #1 is victorious whereas the other armies are decimated.

- Thus the mathematical criteria $k_1 < a_1$, $a_2 + \frac{k_{21}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - (r_{23} + k_2) < 0$, and

$$\frac{k_{31}}{b_1} \ln\left(\frac{k_1}{a_1}\right) - k_3 < 0 \text{ must be maintained through the entire duration of the}$$

conflict.

- Any violation of the above criteria will change the victorious outcome.
- The basic strategy requirement is to ensure that the search has a higher rate of supply. Also that rate constant a_1 is greater than k_1 .

5.3 Summary, Discussions, Conclusions of the Generalized Logistic Model

The generalized logistic model has the form:

$$\begin{cases} \dot{x}_1 = (\alpha_1 - \beta_1 x_1)x_1 - k_{12}x_1x_2 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2 \\ \dot{x}_2 = (\alpha_2 - \beta_2 x_2)x_2 - k_{21}x_2x_1 - r_{23}x_2 - k_{23}x_2x_3 - k_2x_2 - k_{22}x_2^2 - r_{21}x_2 \\ \dot{x}_3 = r_{23}x_2 - k_{31}x_3x_1 - k_{32}x_3x_2 - k_3x_3 - k_{33}x_3^2 \end{cases} \quad 4.3.2a$$

$$\begin{cases} x_1(t=0) = x_{10} \\ x_2(t=0) = x_{20} \\ x_3(t=0) = x_{30} \end{cases} \quad 4.3.2b$$

In particular the rest points are:

$$E_1 = [0, 0, 0]$$

$$E_2 = \left[\frac{1}{\beta_1} \ln \left(\frac{k_1}{\alpha_1} \right), 0, 0 \right]$$

Military Tactic Interpretation on $E_1 = [0, 0, 0]$

“The armies will fight to Mutual Assured Destruction” or mutual annihilation if :

$$C_1 < \frac{k_1}{\beta_1} \text{ and } \alpha_2 < r_{23} + k_2 + r_{21}, \text{ where } C_1 = \frac{\alpha_1}{\beta_1}.$$

In particular C_1 is the logistic maximum for army # 1.

This outcome is analogous to the one observed if the surge rate function is the input function for army # 1; but the rate constants have different interpretations.

- It should be noted that the armies may not have to fight to the bitter end of *zero-end result*, because one of the armies may capitulate before that occur
- To avoid the above scenario, the expeditionary army #1 has to change at least one of the above criteria. In particular, the army #1 must effect a surge such that $\alpha_1 > k_1$.
- Thus the logistic rate constant α_1 must be greater than the rate constant k_1 for non-combat loss of troops.
- Another possibility is to minimize troop loss due to non-combat.

The constant k_1 can be minimized by giving incentive to the army #1, to avoid desertion or protect them from non-combat accidental deaths, or counsel them from suicide. Also it is possible to reduce the number of tours to prevent fatigue-related accidental death or desertion.

Military Tactic Interpretation on $E_3 = \left[\frac{\alpha_1 - k_1}{\beta_1 + k_{11}}, 0, 0 \right]$

The rest point $E_3 = \left[\frac{\alpha_1 - k_1}{\beta_1 + k_{11}}, 0, 0 \right]$ is different from the rest point $E_1 = [0, 0, 0]$ If:

$$\alpha_1 > k_1.$$

Also $\alpha_1 > k_1$ is a necessary but not sufficient condition for the existence of this rest point

- If the above criterion holds then army #1 is victorious whereas the other armies are decimated.

Thus the mathematical criteria:

$$\alpha_1 < k_1, \quad a_2 - k_{21} \frac{\alpha_1 - k_1}{\beta_1 + k_1} < r_{23} + k_2 + r_{21}, \quad \text{and} \quad k_{31} \frac{k_1 - \alpha_1}{\beta_1 + k_1} < k_3$$

must be maintained through the entire duration of the conflict.

- Any violation of the above criteria will change the victorious outcome.

5.4 Future Research

Due to the fact that surge never occurs at the beginning of the war, the mathematical model to be used is the time delay model.

$$\dot{x}_1 = S_1 - k_{12}x_1x_2 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2, \text{ where } S_1 = a_1x_1(t - \tau_1)e^{-b_1x_1(t - \tau_1)}$$

where τ_1 denoted the time at which the surge starts.

Thus

$$\dot{x}_1 = a_1x_1(t - \tau_1)e^{-b_1x_1(t - \tau_1)} - k_{12}x_1x_2 - k_{13}x_1x_3 - k_1x_1 - k_{11}x_1^2 + r_{21}x_2$$

$$\dot{x}_2 = S_2 - K_{21}x_2x_1 - r_{23}x_2 - K_{23}x_2x_3 - k_2x_2 - K_{22}x_2^2 - r_{21}x_2$$

$$\dot{x}_3 = r_{23}x_2 - K_{31}x_3x_1 - k_{32}x_3x_2 - k_3x_3 - k_{33}x_3^2 + k_{33}x_3^2 - r_{21}x_2$$

and

$$S_1 = a_1x_1e^{-b_1x_1}u(t - \tau_1) \text{ with } u = 0 \text{ whenever } t < \tau_1, \text{ and } u = 1 \text{ whenever } t \geq \tau_1$$

Thus

$$S_1 = \begin{cases} 0, & t < \tau_1 \\ a_1x_1e^{-b_1x_1}, & t \geq \tau_1 \end{cases}$$

Hence for $t < \tau_1$, we have:

The time at which the war begins is denoted by t_0 , so the next notation is applied :

$$x_1(t_0) = x_{10}$$

$$x_2(t_0) = x_{20}$$

$$x_3(t_0) = x_{30}$$

$$x_1(t \in [-\tau, t_0)) = \phi_1$$

$$x_2(t \in [-\tau, t_0)) = \phi_2$$

$$x_3(t \in [-\tau, t_0)) = \phi_3$$

$$\phi_i \in C([-\tau, 0])$$

6.0 BIBLIOGRAPHY

- Alligod, Kathleen Tim D. Sauer, James A. Yorke. Chaos: An Introduction to Dynamical Systems. New York: Springer, 1996. 58.
- Amann, Herbert. Ordinary Differential Equations. New York: Walter de Gruyter, 1990. 263-265.
- Bath, U. Narayan, and Gregory K. Miller. Elements of Applied Stochastic Processes. New Jersey: John Wiley & Sons, 2002. 304.
- Clausewitz, Carl Von. Principles of War. Trans. Hans W. Gatzke. Ed. Hans W. Gatzke. New York: Dover Publications, 2003. 53-58.
- Eggenberger, David. An Encyclopedia of Battles: Accounts of Over 1,560 Battles from 1479 B.C. to the Present. New York: Dover Publications, 2007. 32.
- Hofbauer, Josef, and Carl Sigmund. The Theory of Evolutions and Dynamical Systems. London: Cambridge University Press, 1991. 53.
- Nani, F. A Mathematical Treatise on Warfare. (preprint). Edmonton: University of Alberta, 2002.
- Przemieniecki, J.S., and Robert M. Rogers. Mathematical Methods in Defense Analyses. Reston: AIAA Education Series, 2000. 83.
- Tzu, Sun. The Art of War. Trans. Lionel Giles. New York: Dover Publications, 2002. 55.